# MATH 416 Take-Home Exam 2
## Due 11:59pm Thursday, 21 May 2020

Be sure to show all work and to make your reasoning clear.

1. [20] Consider the function $f(x) = 1/\sqrt{1+x^2}$ over $\mathbb{R}$.
   a. [8] Compute the polynomial of degree at most 7 that interpolates the values $f(x)$ at the uniform nodes $\{-7, -5, -3, -1, 1, 3, 5, 7\}$.
   b. [8] Compute the polynomial of degree at most 7 that interpolates the values $f(x)$ at the Chebyshev nodes $\{8r \,:\, T_8(r) = 0\}$.
   c. [4] Plot $f(x)$ and these two interpolants over the interval $[-8, 8]$. Which interpolant gives a better approximation to $f(x)$ over $[-8, 8]$? Why?

2. [20] Let $\phi(t) = \mathrm{hat}(t)$, where the "hat" function is defined over $\mathbb{R}$ by

$$\mathrm{hat}(t) = \begin{cases} 1 - |t| & \text{for } t \in (-1, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This function satisfies the *interpolation condition*,

$$\phi(0) = 1\,, \qquad \phi(k) = 0 \quad \text{for every } k \in \mathbb{Z} - \{0\}\,.$$

Define $\phi_k : \mathbb{R} \to \mathbb{R}$ by $\phi_k(t) = \phi(t - k)$. Let $\{c_k\}_{k\in\mathbb{Z}}$ be any real sequence over $\mathbb{Z}$. For every $m, n \in \mathbb{Z}$ with $m \leq n$ define $u_{mn} : \mathbb{R} \to \mathbb{R}$ by

$$u_{mn}(t) = \sum_{k=m}^{n} c_k\, \phi_k(t)\,.$$

The set $\{u_{mn} \,:\, m, n \in \mathbb{Z}\,,\, m \leq n\}$ has the *Cauchy property* with respect to a norm $\|\cdot\|$ if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for every $m, n \in \mathbb{Z}$ with $m \leq n$ we have

$$n \leq -N_\epsilon \quad \text{or} \quad N_\epsilon \leq m \qquad \Longrightarrow \qquad \|u_{mn}\| < \epsilon\,.$$

Consider the $L^4(\mathbb{R})$ norm defined by

$$\|v\|_{L^4(\mathbb{R})} = \left( \int_{\mathbb{R}} |v(t)|^4 \,\mathrm{d}t \right)^{\frac{1}{4}}\,.$$

   a. [4] Evaluate

$$\|u_{mn}\|_{L^4(\mathbb{R})}^4 = \int_{\mathbb{R}} |u_{mn}(t)|^4 \,dt\,.$$

   b. [8] Show that

$$\tfrac{1}{5} \sum_{k=m}^{n} |c_k|^4 \leq \|u_{mn}\|_{L^4(\mathbb{R})}^4 \leq \sum_{k=m}^{n} |c_k|^4\,.$$

   c. [8] Prove that $\{u_{mn} \,:\, m, n \in \mathbb{Z}\,,\, m \leq n\}$ has the Cauchy property with respect to the $L^4(\mathbb{R})$ norm if and only if

$$\sum_{k\in\mathbb{Z}} |c_k|^4 < \infty\,.$$

3. [20] The Haar wavelet function $\psi : \mathbb{R} \to \mathbb{R}$ is

$$\psi(t) = \begin{cases} 1 & \text{for } t \in [0, \frac{1}{2})\,, \\ -1 & \text{for } t \in [\frac{1}{2}, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

It has a primitive $\Psi : \mathbb{R} \to \mathbb{R}$ given by

$$\Psi(t) = \begin{cases} \min\{t, 1-t\} & \text{for } t \in (0, 1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For each $j, k \in \mathbb{Z}$ define $\psi_{jk} : \mathbb{R} \to \mathbb{R}$ and $\Psi_{jk} : \mathbb{R} \to \mathbb{R}$ by

$$\psi_{jk}(t) = 2^{\frac{j}{2}} \psi(2^j t - k)\,, \qquad \Psi_{jk}(t) = 2^{-\frac{j}{2}} \Psi(2^j t - k)\,.$$

Let $S \subset L^2([0,1])$ be given by

$$S = \big\{ \psi_{jk} \,:\, j \in \{0, 1, \cdots\}\,,\, k \in \{0, 1, \cdots, 2^j - 1\} \big\}\,.$$

Problem 1 of Homework 10 showed that $S$ is an orthonormal set in $L^2([0,1])$ that is orthogonal to every constant function. For every $J \in \mathbb{Z}_+$ let $\mathcal{P}_J : L^2([0,1]) \to L^2([0,1])$ be the orthogonal projection given by

$$\mathcal{P}_J u(t) = \langle 1\,,\, u \rangle + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle \psi_{jk}\,,\, u \rangle\, \psi_{jk}(t) \qquad \text{for every } u \in L^2([0,1])\,.$$

Let $b \in (0,1)$ and set $v(t) = \chi_{[0,b)}(t)$ where

$$\chi_{[0,b)}(t) = \begin{cases} 1 & \text{if } t \in [0, b)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

a. [8] Show for every $J \in \mathbb{Z}_+$ that

$$\mathcal{P}_J v(t) = b + \sum_{j=0}^{J-1} \min\big\{2^j b - \lfloor 2^j b \rfloor\,,\, \lceil 2^j b \rceil - 2^j b\big\}\, \psi\big(2^j t - \lfloor 2^j b \rfloor\big)\,,$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the "floor" and "ceiling" functions, which are defined for every $x \in \mathbb{R}$ by

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} \,:\, k \le x\}\,, \qquad \lceil x \rceil = \min\{k \in \mathbb{Z} \,:\, x \le k\}\,.$$

b. [8] Use induction on $J$ to prove for every $J \in \mathbb{Z}_+$ that

$$\mathcal{P}_J v(t) = \begin{cases} 1 & \text{for } t \in [0, \underline{b}_J)\,, \\ 2^J b - \lfloor 2^J b \rfloor & \text{for } t \in [\underline{b}_J, \overline{b}_J)\,, \\ 0 & \text{for } t \in [\overline{b}_J, 1)\,, \end{cases}$$

where $\underline{b}_J = \lfloor 2^J b \rfloor / 2^J$ and $\overline{b}_J = \lceil 2^J b \rceil / 2^J$.

c. [4] Show for every $J \in \mathbb{Z}_+$ that

$$\|\mathcal{P}_J v - v\|^2_{L^2([0,1])} = \frac{\big(\lceil 2^J b \rceil - 2^J b\big)\big(2^J b - \lfloor 2^J b \rfloor\big)}{2^J} \le \frac{1}{2^{J+2}}\,.$$

4. [20] The Haar scaling function $\phi : \mathbb{R} \to \mathbb{R}$ and wavelet function $\psi : \mathbb{R} \to \mathbb{R}$ are

$$\phi(t) = \begin{cases} 1 & \text{for } t \in [0,1)\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad \psi(t) = \begin{cases} 1 & \text{for } t \in [0,\frac{1}{2})\,, \\ -1 & \text{for } t \in [\frac{1}{2},1)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

They satisfy the two-scale relations

$$\phi(t) = \phi(2t) + \phi(2t-1)\,, \qquad \psi(t) = \phi(2t) - \phi(2t-1)\,.$$

For every $j, k \in \mathbb{Z}$ define $\phi_{jk} : \mathbb{R} \to \mathbb{R}$ and $\psi_{jk} : \mathbb{R} \to \mathbb{R}$ by

$$\phi_{jk}(t) = 2^{\frac{j}{2}} \phi(2^j t - k)\,, \qquad \psi_{jk}(t) = 2^{\frac{j}{2}} \psi(2^j t - k)\,.$$

For every $j \in \mathbb{Z}$ define the subspaces $V_j$ and $W_j$ by

$$V_j = \overline{\text{span}}\{\phi_{jk} \,:\, k \in \mathbb{Z}\}\,, \qquad W_j = \overline{\text{span}}\{\psi_{jk} \,:\, k \in \mathbb{Z}\}\,.$$

a. [8] Show for every $j \in \mathbb{Z}$ that $V_j$ and $W_j$ are orthogonal subspaces.

b. [8] Show for every $j \in \mathbb{Z}$ that

$$V_{j+1} = V_j + W_j = \big\{ v + w \,:\, v \in V_j\,,\, w \in W_j \big\}\,.$$

c. [4] Show for every $j \in \mathbb{Z}_+$ that

$$V_{j+1} = V_0 + W_0 + \cdots + W_j\,.$$

5. [20] Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform given by

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}} e^{-i2\pi\xi t} u(t)\, \mathrm{d}t \qquad \text{for every } u \in L^2(\mathbb{R})\,.$$

Let $\psi : \mathbb{R} \to \mathbb{R}$ be the function given by

$$\psi(t) = 2\,\text{sinc}(2t) - \text{sinc}(t)\,.$$

For every $j, k \in \mathbb{Z}$ define $\psi_{jk} : \mathbb{R} \to \mathbb{R}$ by $\psi_{jk}(t) = 2^{\frac{j}{2}} \psi(2^j t - k)$.

a. [4] Compute $\mathcal{F}\psi(\xi)$. You can use the fact that

$$\mathcal{F}\text{sinc}(\xi) = \begin{cases} 1 & \text{for } |\xi| < \frac{1}{2}\,, \\ \frac{1}{2} & \text{for } |\xi| = \frac{1}{2}\,, \\ 0 & \text{for } |\xi| > \frac{1}{2}\,. \end{cases}$$

b. [4] Compute

$$\int_0^\infty \frac{|\mathcal{F}\psi(\xi)|^2}{\xi}\, \mathrm{d}\xi\,.$$

c. [4] For every $j, k \in \mathbb{Z}$ compute $\mathcal{F}\psi_{jk}(\xi)$.

d. [8] Show that $\{\psi_{jk} \,:\, j, k \in \mathbb{Z}\}$ is an orthonormal set in $L^2(\mathbb{R})$.