MATH 416 Take-Home Exam 1 Solutions
Due 11:00am Friday, 3 April 2020

1. [15] Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denote the usual Euclidean basis given by

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be given by

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 4 \\ 7 \\ -4 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix}.$$

a. [6] Show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ equipped with the usual Euclidean inner product, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{y}$.

b. [9] Express each member of the Euclidean basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as a linear combination of $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

Solution (a). Direct calculations yield

$$\begin{aligned} \mathbf{b}_1^{\mathrm{T}}\mathbf{b}_2 &= 1 \cdot 4 + 4 \cdot 7 + 8 \cdot (-4) = 0, & \mathbf{b}_1^{\mathrm{T}}\mathbf{b}_1 &= 1^2 + 4^2 + 8^2 = 81, \\ \mathbf{b}_2^{\mathrm{T}}\mathbf{b}_3 &= 4 \cdot 8 + 7 \cdot (-4) + (-4) \cdot 1 = 0, & \mathbf{b}_1^{\mathrm{T}}\mathbf{b}_1 &= 4^2 + 7^2 + (-4)^2 = 81, \\ \mathbf{b}_3^{\mathrm{T}}\mathbf{b}_1 &= 8 \cdot 1 + (-4) \cdot 4 + 1 \cdot 8 = 0, & \mathbf{b}_3^{\mathrm{T}}\mathbf{b}_3 &= 1^2 + (-4)^2 + 8^2 = 81. \end{aligned}$$

Because these nonzero vectors are orthogonal, they are linearly independent. Because $\mathbb{R}^3$ has dimension three, the linearly independent vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ will be a basis. Therefore $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. □

Solution (b). Because $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthogonal basis, for every $\mathbf{x} \in \mathbb{R}^3$ we have

$$\mathbf{x} = \frac{\mathbf{b}_1^{\mathrm{T}}\mathbf{x}}{\mathbf{b}_1^{\mathrm{T}}\mathbf{b}_1}\,\mathbf{b}_1 + \frac{\mathbf{b}_2^{\mathrm{T}}\mathbf{x}}{\mathbf{b}_2^{\mathrm{T}}\mathbf{b}_2}\,\mathbf{b}_2 + \frac{\mathbf{b}_3^{\mathrm{T}}\mathbf{x}}{\mathbf{b}_3^{\mathrm{T}}\mathbf{b}_3}\,\mathbf{b}_3 = \frac{\mathbf{b}_1^{\mathrm{T}}\mathbf{x}}{81}\,\mathbf{b}_1 + \frac{\mathbf{b}_2^{\mathrm{T}}\mathbf{x}}{81}\,\mathbf{b}_2 + \frac{\mathbf{b}_3^{\mathrm{T}}\mathbf{x}}{81}\,\mathbf{b}_3.$$

Simply applying this formula to $\mathbf{x} = \mathbf{e}_1$, $\mathbf{x} = \mathbf{e}_2$, and $\mathbf{x} = \mathbf{e}_3$ yields

$$\mathbf{e}_1 = \tfrac{1}{81}\mathbf{b}_1 + \tfrac{4}{81}\mathbf{b}_2 + \tfrac{8}{81}\mathbf{b}_3, \quad \mathbf{e}_2 = \tfrac{4}{81}\mathbf{b}_1 + \tfrac{7}{81}\mathbf{b}_2 - \tfrac{4}{81}\mathbf{b}_3, \quad \mathbf{e}_3 = \tfrac{8}{81}\mathbf{b}_1 - \tfrac{4}{81}\mathbf{b}_2 + \tfrac{1}{81}\mathbf{b}_3.$$

□

Alternative Solution (b). Let $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_1 & \mathbf{b}_1 \end{pmatrix}$. Then we see from part (a) that

$$\mathbf{B}^{\mathrm{T}}\mathbf{B} = \begin{pmatrix} 1 & 4 & 8 \\ 4 & 7 & -4 \\ 8 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 8 \\ 4 & 7 & -4 \\ 8 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} = 81\mathbf{I}.$$

Because $\mathbf{B}^{\mathrm{T}}\mathbf{B} = 81\mathbf{I}$, we see that $\mathbf{B}^{-1} = \frac{1}{81}\mathbf{B}^{\mathrm{T}}$, whereby

$$\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_1 & \mathbf{e}_1 \end{pmatrix} = \mathbf{I} = \mathbf{B}\mathbf{B}^{-1} = \mathbf{B}\tfrac{1}{81}\mathbf{B}^{\mathrm{T}} = \tfrac{1}{81}\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_1 & \mathbf{b}_1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 8 \\ 4 & 7 & -4 \\ 8 & -4 & 1 \end{pmatrix}.$$

The result can be read off from this. □

Remark. This result could also be found by using row reduction to compute $\mathbf{B}^{-1}$. However, this approach does not leverage the information from part (a).

2. [15] Let $\mathbf{A} \in \mathbb{R}^{3\times 4}$ be given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

Define the linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^4$. Equip $\mathbb{R}^4$ with the usual Euclidean inner product. Equip $\mathbb{R}^3$ with the inner product defined by

$$\langle \mathbf{x}\,,\,\mathbf{y}\rangle = \mathbf{x}^{\mathrm{T}}\mathbf{H}\mathbf{y} \qquad \text{for every } \mathbf{x}, \mathbf{y} \in \mathbb{R}^3\,,$$

where $\mathbf{H}$ is the diagonal matrix

$$\mathbf{H} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}\,.$$

Compute $T^* : \mathbb{R}^3 \to \mathbb{R}^4$, the adjoint of T, with respect to these inner products.

Solution. Denote the inner products on $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively by

$$\begin{aligned} \langle \mathbf{x}\,,\,\mathbf{y}\rangle_3 &= \mathbf{x}^{\mathrm{T}}\mathbf{H}\mathbf{y} \qquad \text{for every } \mathbf{x}, \mathbf{y} \in \mathbb{R}^3\,, \\ \langle \mathbf{x}\,,\,\mathbf{y}\rangle_4 &= \mathbf{x}^{\mathrm{T}}\mathbf{y} \qquad\quad \text{for every } \mathbf{x}, \mathbf{y} \in \mathbb{R}^4\,. \end{aligned}$$

Then the adjoint mapping $T^* : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by the relation

$$\langle T^*(\mathbf{x})\,,\,\mathbf{y}\rangle_4 = \langle \mathbf{x}\,,\,T(\mathbf{y})\rangle_3 \qquad \text{for every } \mathbf{x} \in \mathbb{R}^3 \text{ and } \mathbf{y} \in \mathbb{R}^4\,.$$

But for every $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^4$ we have

$$\langle \mathbf{x}\,,\,T(\mathbf{y})\rangle_3 = \langle \mathbf{x}\,,\,\mathbf{A}\mathbf{y}\rangle_3 = \mathbf{x}^{\mathrm{T}}\mathbf{H}\mathbf{A}\mathbf{y} = \big(\mathbf{A}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}\mathbf{x}\big)^{\mathrm{T}}\mathbf{y} = \big(\mathbf{A}^{\mathrm{T}}\mathbf{H}\mathbf{x}\big)^{\mathrm{T}}\mathbf{y} = \langle \mathbf{A}^{\mathrm{T}}\mathbf{H}\mathbf{x}\,,\,\mathbf{y}\rangle_4\,.$$

Therefore

$$\langle T^*(\mathbf{x})\,,\,\mathbf{y}\rangle_4 = \langle \mathbf{A}^{\mathrm{T}}\mathbf{H}\mathbf{x}\,,\,\mathbf{y}\rangle_4 \qquad \text{for every } \mathbf{x} \in \mathbb{R}^3 \text{ and } \mathbf{y} \in \mathbb{R}^4\,.$$

We read off from this that

$$T^*(\mathbf{x}) = \mathbf{A}^{\mathrm{T}}\mathbf{H}\mathbf{x} \qquad \text{for every } \mathbf{x} \in \mathbb{R}^3\,.$$

Therefore the adjoint mapping $T^* : \mathbb{R}^3 \to \mathbb{R}^4$ is given by $T^*(\mathbf{x}) = \mathbf{A}^*\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^3$ where $\mathbf{A}^* \in \mathbb{R}^{4\times 3}$ is given by

$$\mathbf{A}^* = \mathbf{A}^{\mathrm{T}}\mathbf{H} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 10 \\ 6 & -2 & 15 \\ -3 & 4 & 0 \\ 0 & 2 & 5 \end{pmatrix}\,.$$

$\square$

Alternative Solution. More generally, if $\mathbf{A} \in \mathbb{R}^{n\times m}$ and the linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ is given by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^m$, and if $\mathbb{R}^m$ and $\mathbb{R}^n$ are respectively equipped with the inner products

$$\langle \mathbf{x}\,,\,\mathbf{y}\rangle_m = \mathbf{x}^{\mathrm{T}}\mathbf{G}\mathbf{y} \text{ for every } \mathbf{x}, \mathbf{y} \in \mathbb{R}^m\,, \qquad \langle \mathbf{x}\,,\,\mathbf{y}\rangle_n = \mathbf{x}^{\mathrm{T}}\mathbf{H}\mathbf{y} \text{ for every } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n\,,$$

where $\mathbf{G} \in \mathbb{R}^{m\times m}$ and $\mathbf{H} \in \mathbb{R}^{n\times n}$ are positive definite, then the adjoint map with respect to these inner products is $T^* : \mathbb{R}^n \to \mathbb{R}^m$ is given by $T^*(\mathbf{x}) = \mathbf{A}^*\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{A}^* = \mathbf{G}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{H}$. If we apply this formula with $m = 4$, $n = 3$, $\mathbf{G} = \mathbf{I}$, $\mathbf{A}$ given above, and $\mathbf{H}$ given above then we obtain the result. $\square$

3. [15] For every $n \in \mathbb{Z}_+$ let P^n denote all polynomials with real coefficients of degree at most n. Consider the mapping $S : P^n \to P^n$ given by

$$S(p)(t) = (1+t)\,p''(t) + t\,p'(t) - p(t)\,, \qquad \text{for every } p \in P^n\,.$$

Give the matrix representation of S with respect to the basis $\{t^k\}_{k=0}^n$ for $n = 3$.

Solution. Let $\mathbf{a}_k$ be the vector representation of $S(t^k)$. Because

$$S(t^0) = -t^0\,, \qquad S(t^1) = 0\,, \qquad S(t^2) = 2t^0 + 2t^1 + t^2\,, \qquad S(t^3) = 6t^1 + 6t^2 + 2t^3\,,$$

we see that

$$\mathbf{a}_0 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{a}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 6 \\ 6 \\ 0 \end{pmatrix}.$$

The matrix representation of S with respect to the basis $\{t^k\}_{k=0}^3$ is then

$$\mathbf{A}_S = \begin{pmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

□

Alternative Solution. By direct calculation we see that

$$S(t^0) = -t^0\,, \qquad S(t^1) = 0\,, \qquad S(t^2) = 2t^0 + 2t^1 + t^2\,, \qquad S(t^3) = 6t^1 + 6t^2 + 2t^3\,.$$

Hence, if $p(t) = c_0 t^0 + c_1 t^1 + c_2 t^2 + c_3 t^3$ then

$$S(p)(t) = (-c_0 + 2c_2)t^0 + (2c_2 + 6c_3)t^1 + (c_2 + 6c_3)t^2 + 2c_3 t^3\,,$$

whereby we see that

$$\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \mapsto \begin{pmatrix} -c_0 + 2c_2 \\ 2c_2 + 6c_3 \\ c_2 + 6c_3 \\ 2c_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Therefore the matrix representation of S with respect to the basis $\{t^k\}_{k=0}^3$ is

$$\mathbf{A}_S = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

□

Remark. Alternatively, the rows of $\mathbf{A}_S$ can be read off from the coefficients in the expression for $S(p)(t)$ given above.

4. [15] Let $L > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic such that $f(x) = x$ for $x \in [-L, L)$.
   a. [10] Compute the coefficients $\{b_k\}$ in its sine expansion
$$f(x) = \sum_{k=1}^{\infty} b_k \sin\Big(k\frac{\pi}{L}x\Big) \,.$$
   b. [5] Determime whether or not
$$\sum_{k=1}^{n} |b_k| \qquad \text{converges}\,.$$

Remark. Odd symmetry implies that
$$\int_{-L}^{L} \cos\Big(k\frac{\pi}{L}x\Big)\, f(x)\,\mathrm{d}x = \int_{-L}^{L} \cos\Big(k\frac{\pi}{L}x\Big)\, x\,\mathrm{d}x = 0 \quad \text{for every } k \in \mathbb{N}\,.$$
Therefore the Fourier expansion of f will contain only sine terms.

Solution (a). The coefficients in the sine expansion of f are given by
$$\begin{aligned} b_k &= \frac{1}{L}\int_{-L}^{L} \sin\Big(k\frac{\pi}{L}x\Big)\, f(x)\,\mathrm{d}x = \frac{1}{L}\int_{-L}^{L} \sin\Big(k\frac{\pi}{L}x\Big)\, x\,\mathrm{d}x \\ &= -\frac{\cos\Big(k\frac{\pi}{L}x\Big)}{k\pi}\, x\,\Bigg|_{-L}^{L} + \frac{1}{k\pi}\int_{-L}^{L} \cos\Big(k\frac{\pi}{L}x\Big)\,\mathrm{d}x \\ &= -\frac{\cos(k\pi)}{k\pi}L - \frac{\cos(k\pi)}{k\pi}L + 0 = (-1)^{k+1}\frac{2L}{k\pi}\,. \end{aligned}$$
□

Solution (b). Because $|b_k| = \frac{2L}{k\pi}$ is comparable to the terms of the harmonic series,
$$\text{the partal sums} \quad \sum_{k=1}^{n} |b_k| \quad \text{diverge as } n \to \infty\,.$$
□

Remark. If the series did converge then the Weierstrass M-Test would imply that the Fourier sine series would converge uniformly to $f(x)$, which would imply that $f(x)$ is continuous. However, $f(x)$ has jump discontinuities, so the series must diverge.

Let $T > 0$. For each $n \in \mathbb{Z}_+$ define the windowing function $w_n : \mathbb{R} \to [0, \infty)$ by
$$w_n(t) = \begin{cases} \dfrac{1 + p_n\big(\cos(\frac{\pi}{T}\,t)\big)}{2} & \text{for } t \in [-T, T]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$
where $p_n(z)$ is the unique odd polynomial determined by
$$p_n'(z) = c_n(1 - z^2)^{n-1}\,, \qquad p(0) = 0\,, \qquad p(1) = 1\,.$$
Each of these windowing functions satisfy the replication condition
$$\sum_{k\in\mathbb{Z}} w_n(t + kT) = 1\,.$$

5. [20] Given any windowing function $w : \mathbb{R} \to [0, \infty)$ that satisfies the replication condition the T-periodization of the localization wf of any $f : \mathbb{R} \to \mathbb{C}$ is given by

$$f_T(t) = \sum_{k\in\mathbb{Z}} w(t + kT)\, f(t + kT)\,.$$

The k^{th} Fourier coefficient of this periodization is

$$\hat{f}_T(k) = \frac{1}{T}\int_0^T \overline{e_k(t)}\, f_T(t)\,\mathrm{d}t = \frac{1}{T}\int_{-T}^T \overline{e_k(t)}\, w(t)\, f(t)\,\mathrm{d}t\,,$$

where $e_k(t) = \exp(ik\frac{2\pi}{T}t)$ for every $k \in \mathbb{Z}$.

Find $\hat{f}_T(k)$ for every $k \in \mathbb{Z}$ when $f(t) = \exp(i\omega t)$ for some $\omega \in \mathbb{R}$ and $w(t) = w_3(t)$. Integration by parts can be avoided by using the Euler identity $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ along with the trig identities

$$\begin{aligned}(\cos(\theta))^3 &= \tfrac{1}{4}\cos(3\theta) + \tfrac{3}{4}\cos(\theta)\,,\\ (\cos(\theta))^5 &= \tfrac{1}{16}\cos(5\theta) + \tfrac{5}{16}\cos(3\theta) + \tfrac{5}{8}\cos(\theta)\,.\end{aligned}$$

Solution. We are asked to compute

$$\hat{f}_T(k) = \frac{1}{T}\int_{-T}^T e^{-ik\frac{2\pi}{T}t} e^{i\omega t} w_3(t)\,\mathrm{d}t\,.$$

Because

$$p_3'(z) = c_3(1 - 2z^2 + z^4)\,, \qquad p_3(0) = 0\,, \qquad p_3(1) = 1\,,$$

we find that $p_3(z) = \frac{15}{8}(z - \frac{2}{3}z^3 + \frac{1}{5}z^5)$, whereby the trig and Euler identites yield

$$\begin{aligned} w_3(t) &= \tfrac{1}{2}\Big[1 + \tfrac{15}{8}\cos(\tfrac{\pi}{T}t) - \tfrac{5}{4}\big(\cos(\tfrac{\pi}{T}t)\big)^3 + \tfrac{3}{8}\big(\cos(\tfrac{\pi}{T}t)\big)^5\Big]\\ &= \tfrac{1}{2}\big[1 + \tfrac{75}{64}\cos(\tfrac{\pi}{T}t) - \tfrac{25}{128}\cos(\tfrac{3\pi}{T}t) + \tfrac{3}{128}\cos(\tfrac{5\pi}{T}t)\big]\\ &= \tfrac{1}{2}\Big[1 + \tfrac{75}{128}\big(e^{i\frac{\pi}{T}t} + e^{-i\frac{\pi}{T}t}\big) - \tfrac{25}{256}\big(e^{i\frac{3\pi}{T}t} + e^{-i\frac{3\pi}{T}t}\big) + \tfrac{3}{256}\big(e^{i\frac{5\pi}{T}t} + e^{-i\frac{5\pi}{T}t}\big)\Big]\,. \end{aligned}$$

Therefore every integral that needs to be computed has the form

$$\frac{1}{T}\int_{-T}^T e^{-i\mu\frac{\pi}{T}t}\,\mathrm{d}t = \int_{-1}^1 e^{-i\mu\pi t}\,\mathrm{d}t = \frac{e^{-i\mu\pi t}}{-i\mu\pi}\bigg|_{-1}^1 = 2\,\frac{\sin(\mu\pi)}{\mu\pi} = 2\,\mathrm{sinc}(\mu)\,.$$

Set $\omega = \frac{2\pi}{T}\eta$ and apply the above formula to

$$\mu = 2(k-\eta)\,, \quad \mu = 2(k-\eta) \mp 1\,, \quad \mu = 2(k-\eta) \mp 3\,, \quad \mu = 2(k-\eta) \mp 5\,,$$

to obtain

$$\begin{aligned} \hat{f}_T(k) = \ \mathrm{sinc}\big(2(k-\eta)\big) &+ \tfrac{75}{128}\big[\,\mathrm{sinc}\big(2(k-\eta)-1\big) + \mathrm{sinc}\big(2(k-\eta)+1\big)\big]\\ &- \tfrac{25}{256}\big[\,\mathrm{sinc}\big(2(k-\eta)-3\big) + \mathrm{sinc}\big(2(k-\eta)+3\big)\big]\\ &+ \tfrac{3}{256}\big[\,\mathrm{sinc}\big(2(k-\eta)-5\big) + \mathrm{sinc}\big(2(k-\eta)+5\big)\big]\,. \end{aligned}$$

□

Remark. Notice that $\hat{f}_T(k) = \delta_{k\eta}$ if $\eta \in \mathbb{Z}$, which is when $f(t) = e^{i\omega t}$ is T-periodic.

6. [20] Let $N \in \mathbb{Z}_+$. Given any windowing function $w : \mathbb{R} \to [0,\infty)$ that satisfies the replication condition with $T = N$, the N-periodization of the localization wf of any $f : \mathbb{Z} \to \mathbb{C}$ is given by

$$f_N(j) = \sum_{k\in\mathbb{Z}} w(j+kN)\, f(j+kN)\,.$$

The k^{th} Fourier coefficient of this periodization is

$$\hat{f}_N(k) = \frac{1}{N}\sum_{j\in\mathbb{Z}_N} \overline{e_k(j)}\, f_N(j) = \frac{1}{N}\sum_{j=-N}^{N} \overline{e_k(j)}\, w(j)\, f(j)\,,$$

where $e_k(j) = \omega_N^{kj}$ for every $k \in \mathbb{Z}_N$ with $\omega_N = \exp(i\frac{2\pi}{N})$.

Let $\hat{f}_N(k)$ for every $k \in \mathbb{Z}_N$ when $f(j) = \exp(i\omega j)$ for some $\omega \in \mathbb{R}$ and $w(j) = w_1(j)$. Set $N = 100$ and use Matlab to plot $\hat{f}_N(k)$ versus k for $\omega = \frac{\pi}{100}$, $\frac{\pi}{200}$, $\frac{\pi}{400}$, and $\frac{\pi}{800}$.

Solution. We are asked to compute

$$\hat{f}_N(k) = \frac{1}{N}\sum_{j=-N}^{N} e^{-i\frac{2\pi}{N}kj}\, e^{i\omega j}\, w_1(j)\,.$$

Because $p_1'(z) = c_1$, $p_1(0) = 0$, and $p_1(1) = 1$, we see that $p_1(z) = z$, whereby the Euler identity yields

$$w_1(j) = \tfrac{1}{2}\left[1 + \cos(\tfrac{\pi}{N}j)\right] = \tfrac{1}{2}\left[1 + \tfrac{1}{2}\big(e^{i\frac{\pi}{N}j} + e^{-i\frac{\pi}{N}j}\big)\right]\,.$$

Because $w_1(\mp N) = 0$, we have

$$\begin{aligned}\hat{f}_N(k) &= \frac{1}{N}\sum_{j=-(N-1)}^{N-1} e^{-i\frac{2\pi}{N}kj}\, e^{i\omega j}\, w_1(j)\\ &= \frac{1}{2N}\sum_{j=-(N-1)}^{N-1} e^{-i\frac{2\pi}{N}kj}\, e^{i\omega j}\left[1 + \tfrac{1}{2}\big(e^{i\frac{\pi}{N}j} + e^{-i\frac{\pi}{N}j}\big)\right]\,.\end{aligned}$$

This sum decomposes into three finite geometric series in the form

$$\sum_{j=-(N-1)}^{N-1} e^{-i\frac{\pi\mu}{N}j} = \frac{e^{i\pi\mu\frac{N-1}{N}} - e^{-i\pi\mu}}{1 - e^{-i\frac{\pi\mu}{N}}} = \frac{\sin(\pi\mu\frac{2N-1}{2N})}{\sin(\frac{\pi\mu}{2N})} = (2N-1)\frac{\operatorname{sinc}(\mu\frac{2N-1}{2N})}{\operatorname{sinc}(\frac{\mu}{2N})}\,.$$

Set $\omega = \frac{2\pi}{N}\eta$ and apply this formula to $\mu = 2(k-\eta)$ and $\mu = 2(k-\eta)\mp 1$ to obtain

$$\begin{aligned}\hat{f}_N(k) = \frac{2N-1}{2N}\Bigg[&\frac{\operatorname{sinc}((k-\eta)\frac{2N-1}{N})}{\operatorname{sinc}(\frac{k-\eta}{N})} + \frac{\operatorname{sinc}((2(k-\eta)-1)\frac{2N-1}{2N})}{2\operatorname{sinc}(\frac{2(k-\eta)-1}{2N})}\\ &+ \frac{\operatorname{sinc}((2(k-\eta)+1)\frac{2N-1}{2N})}{2\operatorname{sinc}(\frac{2(k-\eta)+1}{2N})}\Bigg]\,.\end{aligned}$$

You are asked set $N = 100$ and to plot $\hat{f}_N(k)$ versus k for $\eta = \frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$. $\square$

Remark. Notice that the continuous result is recovered in the limit

$$\lim_{N\to\infty}\hat{f}_N(k) = \operatorname{sinc}\big(2(k-\eta)\big) + \tfrac{1}{2}\operatorname{sinc}\big(2(k-\eta)-1\big) + \tfrac{1}{2}\operatorname{sinc}\big(2(k-\eta)+1\big)\,.$$