MATH 416 Take-Home Exam 1
Due 11:00am Friday, 3 April 2020

1. [15] Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denote the usual Euclidean basis given by

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be given by

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 4 \\ 7 \\ -4 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix}.$$

a. [6] Show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthogonal basis for $\mathbb{R}^3$ equipped with the usual Euclidean inner product, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{y}$.

b. [9] Express each member of the Euclidean basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as a linear combination of $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

2. [15] Let $\mathbf{A} \in \mathbb{R}^{3\times 4}$ be given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

Define the linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^4$. Equip $\mathbb{R}^4$ with the usual Euclidean inner product. Equip $\mathbb{R}^3$ with the inner product defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{H}\mathbf{y} \qquad \text{for every } \mathbf{x}, \mathbf{y} \in \mathbb{R}^3,$$

where $\mathbf{H}$ is the diagonal matrix

$$\mathbf{H} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Compute $T^* : \mathbb{R}^3 \to \mathbb{R}^4$, the adjoint of T, with respect to these inner products.

3. [15] For every $n \in \mathbb{Z}_+$ let P^n denote all polynomials with real coefficients of degree at most n. Consider the mapping $S : P^n \to P^n$ given by

$$S(p)(t) = (1+t)\,p''(t) + t\,p'(t) - p(t), \qquad \text{for every } p \in P^n.$$

Give the matrix representation of S with respect to the basis $\{t^k\}_{k=0}^{n}$ for $n = 3$.

4. [15] Let $L > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic such that $f(x) = x$ for $x \in [-L, L)$.

a. [10] Compute the coefficients $\{b_k\}$ in its sine expansion

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\!\left(k\frac{\pi}{L}x\right).$$

b. [5] Determime whether or not

$$\sum_{k=1}^{n} |b_k| \qquad \text{converges}.$$

Let $T > 0$. For each $n \in \mathbb{Z}_+$ define the windowing function $w_n : \mathbb{R} \to [0, \infty)$ by

$$w_n(t) = \begin{cases} \dfrac{1 + p_n\big(\cos(\frac{\pi}{T}\,t)\big)}{2} & \text{for } t \in [-T, T]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $p_n(z)$ is the unique odd polynomial determined by

$$p_n'(z) = c_n(1 - z^2)^{n-1}\,, \qquad p(0) = 0\,, \qquad p(1) = 1\,.$$

Each of these windowing functions satisfy the replication condition

$$\sum_{k \in \mathbb{Z}} w_n(t + kT) = 1\,.$$

5. [20] Given any windowing function $w : \mathbb{R} \to [0, \infty)$ that satisfies the replication condition the T-periodization of the localization wf of any $f : \mathbb{R} \to \mathbb{C}$ is given by

$$f_T(t) = \sum_{k \in \mathbb{Z}} w(t + kT)\, f(t + kT)\,.$$

The k^{th} Fourier coefficient of this periodization is

$$\hat{f}_T(k) = \frac{1}{T} \int_0^T \overline{e_k(t)}\, f_T(t)\, \mathrm{d}t = \frac{1}{T} \int_{-T}^{T} \overline{e_k(t)}\, w(t)\, f(t)\, \mathrm{d}t\,,$$

where $e_k(t) = \exp(ik\frac{2\pi}{T}t)$ for every $k \in \mathbb{Z}$.

Find $\hat{f}_T(k)$ for every $k \in \mathbb{Z}$ when $f(t) = \exp(i\omega t)$ for some $\omega \in \mathbb{R}$ and $w(t) = w_3(t)$. Integration by parts can be avoided by using the Euler identity $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ along with the trig identities

$$(\cos(\theta))^3 = \tfrac{1}{4}\cos(3\theta) + \tfrac{3}{4}\cos(\theta)\,,$$
$$(\cos(\theta))^5 = \tfrac{1}{16}\cos(5\theta) + \tfrac{5}{16}\cos(3\theta) + \tfrac{5}{8}\cos(\theta)\,.$$

6. [20] Let $N \in \mathbb{Z}_+$. Given any windowing function $w : \mathbb{R} \to [0, \infty)$ that satisfies the replication condition with $T = N$, the N-periodization of the localization wf of any $f : \mathbb{Z} \to \mathbb{C}$ is given by

$$f_N(j) = \sum_{k \in \mathbb{Z}} w(j + kN)\, f(j + kN)\,.$$

The k^{th} Fourier coefficient of this periodization is

$$\hat{f}_N(k) = \frac{1}{N} \sum_{j \in \mathbb{Z}_N} \overline{e_k(j)}\, f_N(j) = \frac{1}{N} \sum_{j=-N}^{N} \overline{e_k(j)}\, w(j)\, f(j)\,,$$

where $e_k(j) = \omega_N^{kj}$ for every $k \in \mathbb{Z}_N$ with $\omega_N = \exp(i\frac{2\pi}{N})$.

Let $\hat{f}_N(k)$ for every $k \in \mathbb{Z}_N$ when $f(j) = \exp(i\omega j)$ for some $\omega \in \mathbb{R}$ and $w(j) = w_1(j)$. Set $N = 100$ and use Matlab to plot $\hat{f}_N(k)$ versus k for $\omega = \frac{\pi}{100}$, $\frac{\pi}{200}$, $\frac{\pi}{400}$, and $\frac{\pi}{800}$.