FORMULAS FOR ROOTS OF POLYNOMIALS

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ABSTRACT. We give algebraic formulas for the roots of polynomials of degree four or less. Specifically, we review the linear and quadratic formulas, give the cubic and quartic formulas, and then give derivations of those formulas. Some history around these formulas is sketched.

1. Introduction

A polynomial $P(x)$ of degree n has the general form

$$P(x) = p_0 x^n + p_1 x^{n-1} + \cdots + p_{n-1} x + p_n \,,$$

where the p_k are complex numbers with $p_0 \neq 0$. The $n+1$ numbers p_k are called the coefficients of $P(x)$. The polynomial $P(x)$ is called *real* whenever each of its coefficients are real. It is called *monic* whenever $p_0 = 1$.

The roots of the polynomial $P(x)$ are the complex solutions x of the equation

$$P(x) = 0 \,.$$

If $x = r$ is a root of $P(x)$ then

$$P(x) = (x - r) Q(x) \,,$$

where $Q(y)$ is a polynomial of degree $n - 1$. In other words, if $x = r$ is a root of $P(x)$ then $(x - r)$ is a factor of $P(x)$. The multiplicity of the root r is the largest integer m such that

$$P(x) = (x - r)^m R(x) \,,$$

where R is a polynomial of degree $n - m$.

The fundamental theorem of algebra states that there exists complex numbers $\{r_k\}_{k=1}^n$ such that

$$P(x) = p_0 \prod_{k=1}^{n} (x - r_k) = p_0 (x - r_1)(x - r_2) \cdots (x - r_n) \,.$$

Each r_k is a root of $P(x)$. Its multiplicity is the number of times it appears in the list $\{r_k\}_{k=1}^n$.

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Given the roots $\{r_k\}_{k=1}^n$ of any polynomial $P(x)$ of degree n, the coefficients of $P(x)$ are found to satisfy

$$
\begin{aligned}
-\frac{p_1}{p_0} &= \sum_{k=1}^{n} r_k &&= r_1 + r_2 + \cdots + r_n \,, \\
\frac{p_2}{p_0} &= \sum_{\substack{k_1,k_2=1\\ k_1<k_2}}^{n} r_{k_1} r_{k_2} &&= r_1 r_2 + r_1 r_3 + r_2 r_3 + \cdots + r_{n-1} r_n \,, \\
-\frac{p_3}{p_0} &= \sum_{\substack{k_1,k_2,k_3=1\\ k_1<k_2<k_3}}^{n} r_{k_1} r_{k_2} r_{k_3} &&= r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + \cdots + r_{n-2} r_{n-1} r_n \,, \\
&\;\;\vdots \\
(-1)^n \frac{p_n}{p_0} &= r_1 r_2 \cdots r_n \,.
\end{aligned}
\tag{1.1}
$$

In particular, when $P(x)$ is monic ($p_0 = 1$) then its coefficients are given in terms of its roots by formulas (1.1). These are sometimes called the *Viète* or *Vieta formulas* because they were first stated in 1591 for the case of positive roots by François Viète [30], whose Latin pen name was Franciscus Vieta. They were known earlier for n up to at least 4. They were extended to general roots in 1629 by Albert Girard [21], who also first conjectured the fundamental theorem of algebra.

Here we want to go the other way. Namely, we want to express the roots of a polynomial as algebraic formulas of its coefficients. By this we mean formulas that involve only a finite number of additions, subtractions, multiplications, divisions, and roots extractions. Such formulas exist for all polynomials up to degree four. For every higher degree there are polynomials for which no such formula exists. Without loss of generality we can consider only monic polynomials. The root of the general monic linear polynomial,

$$P(x) = x + p_1 \,, \tag{1.2}$$

is given by the *linear formula*,

$$x = -p_1 \,. \tag{1.3}$$

The roots of the general monic quadratic polynomial,

$$P(x) = x^2 + p_1 x + p_2 \,, \tag{1.4}$$

are given by the *quadratic formula*, which is usually given in the form

$$x = \frac{-p_1 \pm (p_1^2 - 4p_2)^{\frac{1}{2}}}{2} \,. \tag{1.5}$$

If p_1 and p_2 are real then this formula yields two real roots when $p_1^2 - 4p_2 > 0$, a conjugate pair of roots when $p_1^2 - 4p_2 < 0$, and a double real root when $p_1^2 - 4p_2 = 0$. The formulas for the roots of the general cubic and quartic monic polynomials are more complicated, and consequently less widely known. We give the cubic and quartic formulas in the next section. They will be derived in later sections.

2. Formulas for Roots of Cubics and Quartics

2.1. **Cubic Formula.** Given the general monic cubic polynomial

$$P(x) = x^3 + p_1 x^2 + p_2 x + p_3 \,, \tag{2.1}$$

set

$$a = -\tfrac{1}{3}p_1 \,, \qquad b = -\tfrac{1}{3}P'(a) \,, \qquad c = -\tfrac{1}{2}P(a) \,. \tag{2.2}$$

Next, find the roots $\{r, s\}$ of the monic quadratic polynomial

$$R(z) = z^2 - 2cz + b^3 \,. \tag{2.3}$$

Let ω denote the cube root of 1 given by

$$\omega = e^{i\frac{2\pi}{3}} = -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2} \,.$$

Then the roots of $P(x)$ are given in terms of a, r and s by the *cubic formula:*

$$x = a + \sigma r^{\frac{1}{3}} + \bar{\sigma} s^{\frac{1}{3}} \,, \qquad \text{where } \sigma \in \{1, \omega, \bar{\omega}\} \,. \tag{2.4}$$

Here the cube roots must be taken so that

$$r^{\frac{1}{3}} \, s^{\frac{1}{3}} = b \,. \tag{2.5}$$

The quadratic polynomial $R(z)$ given by (2.3) is called the *resolvent quadratic* of $P(x)$.

We can use the quadratic formula (1.5) to express the roots of $R(z)$ as

$$r = c + (c^2 - b^3)^{\frac{1}{2}} \,, \qquad s = c - (c^2 - b^3)^{\frac{1}{2}} \,.$$

Then the cubic formula (2.4) can be expressed as

$$x = a + \sigma \left(c + (c^2 - b^3)^{\frac{1}{2}} \right)^{\frac{1}{3}} + \bar{\sigma} \left(c - (c^2 - b^3)^{\frac{1}{2}} \right)^{\frac{1}{3}} \,, \quad \text{where } \sigma \in \{1, \omega, \bar{\omega}\} \,, \tag{2.6}$$

and by (2.5) the cube roots must be taken so that

$$\left(c + (c^2 - b^3)^{\frac{1}{2}} \right)^{\frac{1}{3}} \left(c - (c^2 - b^3)^{\frac{1}{2}} \right)^{\frac{1}{3}} = b \,.$$

Formula (2.6) is the *cubic formula* expressed in terms of the parameters a, b, and c. It can be expressed in terms of the coefficients p_1, p_2, and p_3 through (2.2).

2.2. **Quartic Formula.** Given the general monic quartic polynomial

$$P(x) = x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4 \,, \tag{2.7}$$

set

$$a = -\tfrac{1}{4}p_1 \,, \qquad b = -\tfrac{1}{8}P''(a) \,, \qquad c = -\tfrac{1}{8}P'(a) \,, \qquad d = -\tfrac{1}{4}P(a) \,. \tag{2.8}$$

Next, find the roots $\{r, s, t\}$ of the monic cubic polynomial

$$R(z) = z^3 - 2bz^2 + (b^2 + d)z - c^2 \,. \tag{2.9}$$

Then the roots of $P(x)$ are given in terms of a, r, s, and t by the *quartic formula:*

$$x = a + \sigma r^{\frac{1}{2}} + \tau s^{\frac{1}{2}} + \sigma\tau t^{\frac{1}{2}} \,, \qquad \text{where } \sigma, \tau \in \{1, -1\} \,. \tag{2.10}$$

Here the square roots must be taken so that

$$r^{\frac{1}{2}} \, s^{\frac{1}{2}} \, t^{\frac{1}{2}} = c \,. \tag{2.11}$$

The cubic polynomial $R(z)$ given by (2.9) is called the *resolvent cubic* of $P(x)$.

We can use the cubic formula (2.6) to express the roots of $R(z)$ as

$$
\begin{aligned}
r &= \tfrac{2}{3}b + \Big(g + (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}} + \Big(g - (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}} \,, \\
s &= \tfrac{2}{3}b + \omega\Big(g + (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}} + \bar{\omega}\Big(g - (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}} \,, \\
t &= \tfrac{2}{3}b + \bar{\omega}\Big(g + (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}} + \omega\Big(g - (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}} \,,
\end{aligned}
$$

where the parameters f and g are given in terms of b, c, and d by

$$f = -\tfrac{1}{3}R'\big(\tfrac{2}{3}b\big)\,, \qquad g = -\tfrac{1}{2}R\big(\tfrac{2}{3}b\big)\,, \tag{2.12}$$

and the cube roots are taken so that

$$\Big(g + (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}}\Big(g - (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}} = f\,.$$

Then the quartic formula (2.10) for the roots of $P(x)$ can be expressed as

$$
\begin{aligned}
x = a &+ \sigma\left(\tfrac{2}{3}b + \Big(g + (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}} + \Big(g - (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}}\right)^{\frac{1}{2}} \\
&+ \tau\left(\tfrac{2}{3}b + \omega\Big(g + (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}} + \bar{\omega}\Big(g - (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}}\right)^{\frac{1}{2}} \\
&+ \sigma\tau\left(\tfrac{2}{3}b + \bar{\omega}\Big(g + (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}} + \omega\Big(g - (g^2 - f^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}}\right)^{\frac{1}{2}} \,,
\end{aligned}
\tag{2.13}
$$

where $\sigma, \tau \in \{1, -1\}$ and the outer square roots above are taken to be consistent with (2.11). Formula (2.13) is the *quartic formula* expressed in terms of the parameters a, b, f and g. It can be expressed in terms of the parameters a, b, c, and d by using (2.12) to eliminate f and g. That formula can then be expressed in terms of the coefficients p_1, p_2, p_3, and p_4 through (2.8). There is little advantage to doing this. The resulting formulas would not be very illuminating.

Remark. The quadratic, cubic, and quartic formulas have similar structures. For the quadratic formula we set $a = -\frac{1}{2}p_1$ and $b = -P(a)$. For the cubic formula a, b, and c are given by (2.2). For the quartic formula a, b, c, and d are given by (2.8). Then they have the respective forms:

$$
\begin{aligned}
x &= a + \sigma r^{\frac{1}{2}}\,, && \text{where } \sigma \in \{1, -1\}\,; \\
x &= a + \sigma r^{\frac{1}{3}} + \bar{\sigma} s^{\frac{1}{3}}\,, && \text{where } \sigma \in \{1, \omega, \bar{\omega}\}\,; \\
x &= a + \sigma r^{\frac{1}{2}} + \tau s^{\frac{1}{2}} + \sigma\tau t^{\frac{1}{2}}\,, && \text{where } \sigma, \tau \in \{1, -1\}\,.
\end{aligned}
$$

Here respectively

$$
\begin{array}{ccl}
r & \text{is the root of} & R(z) = z - b\,, \\
\{r, s\} & \text{are the roots of} & R(z) = z^2 - 2cz + b^3\,, \\
\{r, s, t\} & \text{are the roots of} & R(z) = z^3 - 2bz^2 + (b^2 + d)z - c^2\,.
\end{array}
$$

Solutions of the cubic and quartic were first published in 1545 by Girolamo Cardano in his book *Ars Magna* [6]. For the cubic he credits Scipione del Ferro, Niccolò Tartaglia (Nicolo Tartalea), and himself. For the quartic he credits his student, Lodovico Ferrari. However, not all solutions were found. Rendering the quartic formula in terms of all the roots of its resolvent cubic was done in 1738 by Leonhard Euler [10]. Our presentation of it parallels that of Euler.

3. Derivations of the Quadratic Formula

Before deriving the cubic and quartic formulas, it is helpful to examine the quadratic formula (1.5) from three perspectives.

The least enlightening perspective is to simply plug it into (1.4) and verify that it yields roots of $P(x)$. This view gives no insight into either the derivation of the formula or why there are usually two roots.

A much better perspective is to understand that the quadratic formula can be derived by first "completing the square" of $P(x)$ as

$$P(x) = \big(x + \tfrac{1}{2}p_1\big)^2 + \big(p_2 - \tfrac{1}{4}p_1^2\big)\,. \tag{3.1}$$

The roots of $P(x)$ must thereby satisfy

$$\big(x + \tfrac{1}{2}p_1\big)^2 = \tfrac{1}{4}p_1^2 - p_2\,.$$

Upon simply taking square roots we see that

$$x + \tfrac{1}{2}p_1 = \pm\big(\tfrac{1}{4}p_1^2 - p_2\big)^{\frac{1}{2}}\,.$$

We then arrive at the quadratic formula (1.5) by this solving for x. The formula yields two roots except when $p_1^2 - 4p_2 = 0$, in which case it is clear from (3.1) that the one root it yields has multiplicity two.

A third perspective that gives greater insight into how the cubic and quartic formulas come about is to view the process of completing the square as a transformation of $P(x)$ into another quadratic polynomial whose roots are easier to find. More precisely, we first define a to be the solution of $P'(a) = 0$, by which

$$a = -\tfrac{1}{2}p_1\,.$$

The Taylor expansion of $P(x)$ at a then yields

$$P(x) = P(a) + \tfrac{1}{2}P''(a)(x-a)^2 = P(a) + (x-a)^2\,.$$

We define the quadratic polynomial $Q(y)$ by

$$Q(y) = P(a+y) = y^2 - b\,, \tag{3.2}$$

where

$$b = -P(a) = \tfrac{1}{4}p_1^2 - p_2\,.$$

The polynomial $Q(y)$ is called the *normal form polynomial* for $P(x)$. It is clear from (3.2) that x_* is a root of $P(x)$ if and only if $x_* = a + y_*$ where y_* is root of $Q(y)$. But the roots of $Q(y)$ are clearly given by $y = \pm b^{\frac{1}{2}}$, so the roots of $P(x)$ are given by

$$x = a \pm b^{\frac{1}{2}}\,. \tag{3.3}$$

This is simply a recasting of the quadratic formula (1.5).

4. Derivation of the Cubic Formula

Here we derive formula (2.6) for the roots of the general monic cubic polynomial,

$$P(x) = x^3 + p_1 x^2 + p_2 x + p_3 \,. \tag{4.1}$$

We will first transformation $P(x)$ into another cubic polynomial that is in a *normal form*. We will then reduce the problem of finding roots of this normal form polynomial to that of finding solutions to an *auxiliary system* of equations. Next, we reduce the problem of solving the auxiliary system to those of finding the roots of a quadratic equation and of taking cube roots.

4.1. **Transformation to Normal Form.**

Here we transformation $P(x)$ into its *normal form polynomial*, another cubic polynomial whose roots are related to those of $P(x)$. The first two derivatives of $P(x)$ are

$$P'(x) = 3x^2 + 2p_1 x + p_2 \,, \qquad P''(x) = 6x + 2p_1 \,.$$

We define a to be the solution of $P''(a) = 0$, which yields

$$a = -\tfrac{1}{3} p_1 \,.$$

The Taylor expansion of $P(x)$ at a then yields

$$\begin{aligned} P(x) &= P(a) + P'(a)(x-a) + \tfrac{1}{6} P'''(a)(x-a)^3 \\ &= P(a) + P'(a)(x-a) + (x-a)^3 \,. \end{aligned}$$

The so-called *normal form polynomial* $Q(y)$ is defined by

$$Q(y) = P(a+y) = y^3 - 3by - 2c \,, \tag{4.2}$$

where

$$b = -\tfrac{1}{3} P'(a) \,, \qquad c = -\tfrac{1}{2} P(a) \,.$$

It follows from (4.2) that

$$P(x) = Q(x-a) \,.$$

It is clear from (4.2) that x_* is a root of $P(x)$ if and only if $x_* = a + y_*$ where y_* is root of $Q(y)$. We thereby reduce the problem of finding the roots of $P(x)$, the general cubic monic polynomial (4.1), to that of finding the roots of $Q(y)$, the normal form polynomial (4.2).

4.2. **Deriving an Auxiliary System.**

The key to seeing how to find the roots of the normal form polynomial $Q(y)$ is the cubic binomial identity

$$(u+v)^3 = u^3 + 3u^2 v + 3uv^2 + v^3 \,,$$

rewritten as

$$(u+v)^3 - 3uv(u+v) - (u^3 + v^3) = 0 \,.$$

This shows that if we can find (u, v) that satisfies the so-called *auxiliary system*,

$$uv = b \,, \qquad u^3 + v^3 = 2c \,, \tag{4.3}$$

then $y = u + v$ is a root of the normal form

$$Q(y) = y^3 - 3by - 2c \,.$$

This reduces the problem of finding roots of $Q(y)$ to that of finding solutions of system (4.3).

We now make two observations. First, if (u, v) is a solution of system (4.3) then so is (v, u). Second, if (u, v) is a solution of system (4.3) then so are $(\omega u, \bar{\omega} v)$ and $(\bar{\omega} u, \omega v)$, where ω is the cube root of 1 given by

$$\omega = e^{i\frac{2\pi}{3}} = -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\,.$$

By putting these observations together we see that each solution (u, v) of system (4.3) represents the set of (up to six) solutions given by

$$\begin{array}{ccc} (u, v)\,, & (\omega u, \bar{\omega} v)\,, & (\bar{\omega} u, \omega v)\,, \\ (v, u)\,, & (\bar{\omega} v, \omega u)\,, & (\omega v, \bar{\omega} u)\,. \end{array} \tag{4.4}$$

Roots of $Q(y)$ are obtained by summing the entries of each solution. The two solutions in each column generate the same root. Therefore three roots of $Q(y)$ are given by

$$y_0 = u + v\,, \qquad y_1 = \omega u + \bar{\omega} v\,, \qquad y_2 = \bar{\omega} u + \omega v\,. \tag{4.5}$$

These yield three roots of $P(x)$ given by

$$x_0 = a + u + v\,, \qquad x_1 = a + \omega u + \bar{\omega} v\,, \qquad x_2 = a + \bar{\omega} u + \omega v\,. \tag{4.6}$$

Remark. It is easy to check that

$$\begin{aligned} \omega u + \bar{\omega} v = \bar{\omega} u + \omega v \qquad &\Longleftrightarrow \qquad u = v\,; \\ u + v = \omega u + \bar{\omega} v \qquad &\Longleftrightarrow \qquad v = \omega u\,; \\ u + v = \bar{\omega} u + \omega v \qquad &\Longleftrightarrow \qquad v = \bar{\omega} u\,. \end{aligned}$$

Therefore the roots of $Q(y)$ given by (4.5) will be distinct if and only if $v \notin \{u, \omega u, \bar{\omega} u\}$. Notice that $v \in \{u, \omega u, \bar{\omega} u\}$ if and only if

$$2c = u^3 + v^3 = 2u^3\,,$$

and

$$b = uv \in \{u^2, \omega u^2, \bar{\omega} u^2\}\,,$$

which holds if and only if $b^3 = u^6 = c^2$. Therefore the formulas (4.5) yield three distinct roots of $Q(y)$ from any one solution (u, v) of the auxiliary system (4.3) if and only if $b^3 \neq c^2$.

4.3. **Equivalence of the Auxiliary System.** Subsection 4.2 showed that if (u, v) is a solution of the auxiliary system (4.3) then the cubic polynomial $Q(y)$ given by (4.2) has roots $\{y_0, y_1, y_2\}$ given by (4.5). Here we show the converse — namely, that if $\{y_0, y_1, y_2\}$ are the roots of $Q(y)$ then there is a solution (u, v) of the auxiliary system (4.3) such that (y_0, y_1, y_2) are given by (4.5). Therefore solving the auxiliary system (4.3) is equivalent to finding the roots of $Q(y)$.

Let $\{y_0, y_1, y_2\}$ be the roots of $Q(y)$. Define (u, v) by

$$u = \tfrac{1}{3}(y_0 + \bar{\omega} y_1 + \omega y_2)\,, \qquad v = \tfrac{1}{3}(y_0 + \omega y_1 + \bar{\omega} y_2)\,. \tag{4.7}$$

Because

$$\begin{pmatrix} 0 \\ u \\ v \end{pmatrix} = \tfrac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \bar{\omega} & \omega \\ 1 & \omega & \bar{\omega} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}\,,$$

and because

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \bar{\omega} & \omega \\ 1 & \omega & \bar{\omega} \end{pmatrix}^{-1} = \tfrac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix}\,,$$

we see that (y_0, y_1, y_2) is given by (4.5).

All that remains is to check that (u,v) defined by (4.7) solves the auxiliary system (4.3). Because $\{y_0, y_1, y_2\}$ are given by (4.5), we see that

$$y_1 + y_2 = -(u+v)\,, \qquad y_1 y_2 = u^2 - uv + v^2\,.$$

Because $\{y_0, y_1, y_2\}$ are the roots of $Q(y)$ we obtain

$$\begin{aligned} Q(y) &= (y-y_0)(y-y_1)(y-y_2) \\ &= y^3 + \big(y_0(y_1+y_2) + y_1 y_2\big)y - y_0 y_1 y_2 \\ &= y^3 + \big(-(u+v)^2 + (u^2 - uv + v^2)\big)y - (u+v)(u^2 - uv + v^2) \\ &= y^3 - 3uvy - (u^3 + v^3)\,. \end{aligned}$$

By comparing this with (4.2) we conclude that (u,v) solves the auxiliary system (4.3).

4.4. **Solving the Auxiliary System.** To show that the auxiliary system (4.3) has a solution, observe that because

$$u^3 + v^3 = 2c\,, \qquad u^3 v^3 = b^3\,,$$

the Vieta formulas (1.1) show that u^3 and v^3 are roots of the *resolvent quadratic*

$$R(z) = z^2 - 2cz + b^3\,.$$

By the quadratic formula we can (without loss of generality) write

$$u^3 = c + (c^2 - b^3)^{\frac{1}{2}}\,, \qquad v^3 = c - (c^2 - b^3)^{\frac{1}{2}}\,. \tag{4.8}$$

We then take cube roots to obtain

$$u = \Big(c + (c^2 - b^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}}\,, \qquad v = \Big(c - (c^2 - b^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}}\,, \tag{4.9}$$

There are choices to be made when taking the square roots in (4.8) and when taking the cube roots in (4.9). Notice that u^3 and v^3 given by (4.8) will satisfy the equation $u^3 + v^3 = 2c$ in the auxiliary system (4.3) no matter what choice is made about the square roots, but u and v given by (4.9) may not satisfy the equation $uv = b$ unless the cube roots are taken carefully. When u and v given by (4.9) are placed into (4.6) we obtain the cubic formula (2.6).

4.5. **Analyzing the Case of Real Coefficients.** When b and c are real there are three cases to consider: $c^2 - b^3 > 0$, $c^2 - b^3 = 0$, and $c^2 - b^3 < 0$. The quantity $c^2 - b^3$ is called the *discriminant* of $Q(y)$.

4.5.1. *Positive Discriminant.* If $c^2 - b^3 > 0$ then set

$$u = \Big(c + (c^2 - b^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}}\,, \qquad v = \Big(c - (c^2 - b^3)^{\frac{1}{2}}\Big)^{\frac{1}{3}}\,, \tag{4.10}$$

where the square roots are taken to be positive and the cube roots are taken to be real. With these conventions we have $v < u$. We then see from (4.5) that $Q(y)$ has one real simple root given by

$$y_0 = u + v\,, \tag{4.11a}$$

and a conjuagate pair of simple roots given by

$$\begin{aligned} y_1 &= \omega u + \bar{\omega} v = -\tfrac{1}{2}(u+v) + i\tfrac{\sqrt{3}}{2}(u-v)\,, \\ y_2 &= \bar{\omega} u + \omega v = -\tfrac{1}{2}(u+v) - i\tfrac{\sqrt{3}}{2}(u-v)\,. \end{aligned} \tag{4.11b}$$

4.5.2. *Zero Discriminant.* If $c^2 - b^3 = 0$ then set

$$u = v = c^{\frac{1}{3}}\,, \tag{4.12}$$

where the cube root is taken to be real. We then see from (4.5) that if $u \neq 0$ then $Q(y)$ has one real simple root given by

$$y_0 = 2u\,, \tag{4.13a}$$

and one real double root given by

$$y_1 = y_2 = -u\,. \tag{4.13b}$$

If $u = 0$ then $Q(y)$ has one real triple root given by $y_0 = y_1 = y_2 = 0$.

4.5.3. *Negative Discriminant.* If $c^2 - b^3 < 0$ then $b > 0$. Set

$$u = \Big(c + i(b^3 - c^2)^{\frac{1}{2}}\Big)^{\frac{1}{3}} = \sqrt{b}e^{i\theta}\,, \qquad v = \bar{u} = \sqrt{b}e^{-i\theta}\,, \tag{4.14}$$

where the square roots are taken to be positive and $\theta \in (0, \frac{\pi}{3})$ is given by

$$\theta = \tfrac{1}{3}\cos^{-1}\left(\frac{c}{b^{\frac{3}{2}}}\right)\,. \tag{4.15}$$

We then see from (4.5) that $Q(y)$ has three real simple roots given by

$$\begin{aligned} y_0 &= u + v && = 2\sqrt{b}\cos(\theta)\,,\\ y_1 &= \omega u + \bar{\omega} v && = 2\sqrt{b}\cos(\theta + \tfrac{2\pi}{3}) = -\sqrt{b}\cos(\theta) - \sqrt{3b}\sin(\theta)\,,\\ y_2 &= \bar{\omega} u + \omega v && = 2\sqrt{b}\cos(\theta - \tfrac{2\pi}{3}) = -\sqrt{b}\cos(\theta) + \sqrt{3b}\sin(\theta)\,. \end{aligned} \tag{4.16}$$

Remark. Because $\theta \in (0, \frac{\pi}{3})$ we have the ordering

$$0 < \theta < \tfrac{\pi}{3} < \tfrac{2\pi}{3} - \theta < \tfrac{2\pi}{3} < \tfrac{2\pi}{3} + \theta < \pi\,.$$

Because cosine is decreasing over $(0, \pi)$, the above inequalities imply that

$$-1 < \cos(\tfrac{2\pi}{3} + \theta) < -\tfrac{1}{2} < \cos(\tfrac{2\pi}{3} - \theta) < \tfrac{1}{2} < \cos(\theta) < 1\,.$$

It follows that

$$2\sqrt{b}\cos(\tfrac{2\pi}{3} + \theta) < 2\sqrt{b}\cos(\tfrac{2\pi}{3} - \theta) < 2\sqrt{b}\cos(\theta)\,.$$

whereby the real roots given by (4.16) are ordered as

$$y_1 < y_2 < y_0\,.$$

Remark. Formulas (4.16) were first derived in 1591 by Viète [30] using a different approach than the one that we took above. There is no simple formula for $\cos(\theta)$ in terms of b and c. Formula (4.15) involves the inverse cosine function, which is not a simple algebraic function. Hence, formulas (4.16) are less satisfying than formulas (4.11) and (4.13).

5. Derivation of the Quartic Formula

Here we derive formula (2.10) for the roots of the general monic quartic polynomial,

$$P(x) = x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 \,. \tag{5.1}$$

We will first transformation $P(x)$ into another quartic polynomial that is in a *normal form*. We will then reduce the problem of finding roots of this normal form polynomial to that of finding solutions to an *auxiliary system* of equations. Next, we reduce the problem of solving the auxiliary system to those of finding the roots of a cubic polynomial and of taking square roots.

5.1. **Transformation to Normal Form.**

Here we transformation $P(x)$ into its *normal form polynomial*, another quartic polynomial whose roots are related to those of $P(x)$. The first three derivatives of $P(x)$ are

$$\begin{aligned} P'(x) &= 4x^3 + 3p_1x^2 + 2p_2x + p_3 \,, \\ P''(x) &= 12x^2 + 6p_1x + 2p_2 \,, \\ P'''(x) &= 24x + 6p_1 \,. \end{aligned}$$

We define a to be the solution of $P'''(a) = 0$, which yields

$$a = -\tfrac{1}{4}p_1 \,.$$

The Taylor expansion of $P(x)$ at a then yields

$$\begin{aligned} P(x) &= P(a) + P'(a)(x-a) + \tfrac{1}{2}P''(a)(x-a)^2 + \tfrac{1}{24}P''''(a)(x-a)^4 \\ &= P(a) + P'(a)(x-a) + \tfrac{1}{2}P''(a)(x-a)^2 + (x-a)^4 \,. \end{aligned}$$

The so-called *normal form polynomial* $Q(y)$ is defined by

$$Q(y) = P(a+y) = y^4 - 4by^2 - 8cy - 4d \,, \tag{5.2}$$

where

$$b = -\tfrac{1}{8}P''(a) \,, \qquad c = -\tfrac{1}{8}P'(a) \,, \qquad d = -\tfrac{1}{4}P(a) \,.$$

It follows from (5.2) that

$$P(x) = Q(x-a) \,.$$

It is clear from (5.2) that x_* is a root of $P(x)$ if and only if $x_* = a + y_*$ where y_* is root of $Q(y)$. We thereby reduce the problem of finding the roots of $P(x)$, the general quartic monic polynomial (5.1), to that of finding the roots $Q(y)$, the normal form polynomial (5.2).

5.2. **Deriving an Auxiliary System.**

The keys to seeing how to find the roots of the normal form polynomial $Q(y)$ are the quadratic and quartic trinomial identities

$$\begin{aligned} (u+v+w)^2 &= u^2 + v^2 + w^2 + 2(uv + vw + uw) \,, \\ (u+v+w)^4 &= (u^2+v^2+w^2)^2 + 4(u^2+v^2+w^2)(uv+vw+uw) \\ &\quad + 8uvw(u+v+w) + 4\big(u^2v^2 + v^2w^2 + u^2w^2\big) \,. \end{aligned}$$

By setting $y = u + v + w$ into $Q(y)$ given by (5.2) we obtain

$$\begin{aligned} Q(y) = {} & (u^2+v^2+w^2)^2 - 4b(u^2+v^2+w^2) \\ & + 4\big((u^2+v^2+w^2) - 2b\big)(uv+vw+uw) \\ & + 8(uvw - c)(u+v+w) + 4\big(u^2v^2 + v^2w^2 + u^2w^2 - d\big) \,. \end{aligned}$$

This shows that if we can find (u, v, w) that satisfies the so-called *auxiliary system*,

$$u^2 + v^2 + w^2 = 2b\,, \qquad uvw = c\,, \qquad u^2v^2 + v^2w^2 + u^2w^2 = d + b^2\,, \tag{5.3}$$

then $y = u + v + w$ is a root of the normal form

$$Q(y) = y^4 - 4by^2 - 8cy - 4d\,.$$

This reduces the problem of finding roots of $Q(y)$ to that of finding solutions of system (5.3).

We now make two observations. First, if (u, v, w) is a solution of system (5.3) then so are (v, w, u), (w, u, v), (v, u, w), (u, w, v), and (w, v, u) — i.e. every permutation of its entries. Second, if (u, v, w) is a solution of system (5.3) then so are $(-u, -v, w)$, $(-u, v, -w)$, and $(u, -v, -w)$. By putting these observations together we see that each solution (u, v, w) of system (5.3) represents the set of (up to twenty four) solutions given by

$$\begin{array}{llll}
(u, v, w)\,, & (u, -v, -w)\,, & (-u, v, -w)\,, & (-u, -v, w)\,, \\
(v, w, u)\,, & (-v, -w, u)\,, & (v, -w, -u)\,, & (-v, w, -u)\,, \\
(w, u, v)\,, & (-w, u, -v)\,, & (-w, -u, v)\,, & (w, -u, -v)\,, \\
(v, u, w)\,, & (-v, u, -w)\,, & (v, -u, -w)\,, & (-v, -u, w)\,, \\
(u, w, v)\,, & (u, -w, -v)\,, & (-u, -w, v)\,, & (-u, w, -v)\,, \\
(w, v, u)\,, & (-w, -v, u)\,, & (-w, v, -u)\,, & (w, -v, -u)\,.
\end{array} \tag{5.4}$$

Roots of $Q(y)$ are obtained by summing the entries of each solution. The six solutions in each column generate the same root. Therefore four roots of $Q(y)$ are given by

$$\begin{aligned}
y_0 &= u + v + w\,, & y_1 &= u - v - w\,, \\
y_2 &= -u + v - w\,, & y_3 &= -u - v + w\,.
\end{aligned} \tag{5.5}$$

These yield four roots of $P(x)$ given by

$$\begin{aligned}
x_0 &= a + u + v + w\,, & x_1 &= a - u + v - w\,, \\
x_2 &= a + u - v - w\,, & x_3 &= a - u - v + w\,.
\end{aligned} \tag{5.6}$$

5.3. **Equivalence of the Auxiliary System.** Subsection 5.2 showed that if (u, v, w) is a solution of the auxiliary system (5.3) then the quartic polynomial $Q(y)$ given by (5.2) has roots $\{y_0, y_1, y_2, y_3\}$ given by (5.5). Here we show the converse — namely, that if $\{y_0, y_1, y_2, y_3\}$ are the roots of $Q(y)$ then there is a solution (u, v, w) of the auxiliary system (5.3) such that (y_0, y_1, y_2, y_3) is given by (5.5). Therefore solving the auxiliary system (5.3) is equivalent to finding the roots of $Q(y)$.

Let $\{y_0, y_1, y_2, y_3\}$ be the roots of $Q(y)$. Define (u, v, w) by

$$\begin{aligned}
u &= \tfrac{1}{4}(y_0 + y_1 - y_2 - y_3)\,, \\
v &= \tfrac{1}{4}(y_0 - y_1 + y_2 - y_3)\,, \\
w &= \tfrac{1}{4}(y_0 - y_1 - y_2 + y_3)\,.
\end{aligned} \tag{5.7}$$

Because

$$\begin{pmatrix} 0 \\ u \\ v \\ w \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

and because

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

it follows that (y_0, y_1, y_2, y_3) is given by (5.5).

All that remains is to check that (u, v, w) defined by (5.7) solves the auxiliary system (5.3). Because $\{y_0, y_1, y_2, y_3\}$ are the roots of $Q(y)$, we see that

$$\begin{aligned} Q(y) &= (y - y_0)(y - y_1)(y - y_2)(y - y_3) \\ &= (y - u - v - w)(y - u + v + w)(y + u - v + w)(y + u + v - w) \\ &= \big((y - u)^2 - (v + w)^2\big)\big((y + u)^2 - (v - w)^2\big) \\ &= \big(y^2 - 2uy + u^2 - v^2 - w^2 - 2vw\big)\big(y^2 + 2uy + u^2 - v^2 - w^2 + 2vw\big) \\ &= y^4 - 2(u^2 + v^2 + w^2)y^2 - 8uvwy + \big((u^2 - v^2 - w^2)^2 - 4v^2w^2\big) \\ &= y^4 - 2(u^2 + v^2 + w^2)y^2 - 8uvwy + \big((u^2 + v^2 + w^2)^2 - 4(u^2v^2 + u^2w^2 + v^2w^2)\big)\,. \end{aligned} \tag{5.8}$$

By comparing this with (5.2) we conclude that (u, v, w) solves

$$u^2 + v^2 + w^2 = 2b\,, \qquad uvw = c\,, \qquad 4(u^2v^2 + u^2w^2 + v^2w^2) - (u^2 + v^2 + w^2)^2 = 4d\,.$$

But this is equivalent to (u, v, w) solving the auxiliary system (5.3).

5.4. **Solving the Auxiliary System.** To show that the auxiliary system (5.3) has a solution, observe that because

$$u^2 + v^2 + w^2 = 2b\,, \qquad u^2v^2 + v^2w^2 + u^2w^2 = d + b^2\,, \qquad u^2v^2w^2 = c^2\,,$$

the Vieta formulas (1.1) show that $\{u^2, v^2, w^2\}$ are the roots of the *resolvent cubic*

$$R(z) = z^3 - 2bz^2 + (d + b^2)z - c^2\,. \tag{5.9}$$

We can use the cubic formula (2.6) to obtain the roots $\{r, s, t\}$ of the cubic $R(z)$. We then take square roots to obtain

$$u = r^{\frac{1}{2}}\,, \qquad v = s^{\frac{1}{2}}\,, \qquad w = t^{\frac{1}{2}}\,. \tag{5.10}$$

There are choices of sign to be made when taking the square roots in (5.10). Notice that (u, v, w) given by (5.10) will satisfy the equations $u^2 + v^2 + w^2 = 2b$ and $u^2v^2 + v^2w^2 + u^2w^2 = d + b^2$ of system (5.3) for any choice of signs, but may not satisfy the equation $uvw = c$ unless the signs are chosen carefully. When u, v, and w given by (5.10) are placed into (5.6) we obtain the quartic formula (2.10).

Remark. It is easy to check that

$$\begin{aligned} u + v + w &= u - v - w & &\iff & v + w &= 0\,; \\ u + v + w &= -u + v - w & &\iff & u + w &= 0\,; \\ u + v + w &= -u - v + w & &\iff & u + v &= 0\,; \\ u - v - w &= -u + v - w & &\iff & u - v &= 0\,; \\ u - v - w &= -u - v + w & &\iff & u - w &= 0\,; \\ -u + v - w &= -u - v + w & &\iff & v - w &= 0\,. \end{aligned}$$

Therefore the roots of $Q(y)$ given by (5.5) will be distinct if and only if

$$\begin{aligned} 0 &\neq (v+w)(u+w)(u+v)(u-v)(u-w)(v-w) \\ &= (v^2-w^2)(u^2-w^2)(u^2-v^2)\,. \end{aligned}$$

Therefore $Q(y)$ given by (5.2) has distinct roots if and only if its resolvent cubic $R(z)$ given by (5.9) has distinct roots.

Remark. When $Q(y)$ has real coefficients and $c \neq 0$ then the Intermediate-Value Theorem implies that $R(z)$ has a positive root because $R(0) = -c^2$ is negative while $R(z)$ becomes positive as $z \to \infty$. Therefore without loss of generality we may take $r > 0$. We can also take $u = r^{\frac{1}{2}} > 0$. Then we see from (5.8) and (5.10) that $Q(y)$ factors as

$$\begin{aligned} Q(y) &= \big(y^2 - 2uy + u^2 - v^2 - w^2 - 2vw\big)\big(y^2 + 2uy + u^2 - v^2 - w^2 + 2vw\big) \\ &= \left(y^2 - 2uy + 2u^2 - 2b - \frac{2c}{u}\right)\left(y^2 + 2uy + 2u^2 - 2b + \frac{2c}{u}\right). \end{aligned}$$

Here we have used the facts that $u^2 + v^2 + w^2 = 2b$ and that $uvw = c$. This shows that $Q(y)$ can be factored into two quadratic factors with real coefficients when $c \neq 0$.

Remark. The above fact was used by Euler [11] in his 1749 algebraic proof of the Fundamental Theorem of Algebra. However there were gaps in his proof. The first satisfactory proof was given by Argand [4] in 1815. This followed a string of unsatisfactory proofs that included efforts by d'Alembert [8] in 1746, Euler [11] in 1749, Foncenex [12, 13] in 1759, Lagrange [24] in 1771, Laplace [26] in 1795, Wood [31] in 1798, and Gauss [16] in his 1799 dissertation. Gauss pointed out gaps in earlier proofs. His own proof had a subtle topological gap that was filled much later. Argand was the first to treat polynomials with complex coefficients. His proof was outlined in a privately published 1806 pamphlet [3]. It was presented as an application of his geometric interpretaion of complex numbers as points in the plane — what we now call the Argand or complex plane. It built upon the proof of d'Alembert. It argues that $|P(x)|$ has a minimizer x_* in the complex plane and then derives a contradiction if $P(x_*) \neq 0$. It was followed by new satisfactory proofs by Gauss [18, 19, 20] in 1815, 1816 and 1849, only the last of which treated complex coefficients. Now there are many good proofs of the Fundamental Theorem of Algebra. For example, it is an easy corollary of the Loiuville Theorem in complex analysis, which was proved in the 1840s by Cauchy and Loiuville [7, 27].

Remark. The combined work of del Ferro, Tartalea, Cardano, and Ferrari on roots of cubics and quartics [6, 29] led to the first systematic development of the complex numbers by Bombelli, which appeared in his 1572 book *Algebra* [5]. This would eventually allowed all roots of those equations to be found. This effort was advanced by Viète [30] in 1591, Girard [21] in 1629, Harriot [22] in 1631, and Descartes [9] in 1637. The next big advance was the rendering the quartic formula in terms of all the roots of its resolvent cubic by Euler [10] in 1738. Moreover, Euler suggested a form that roots of higher degree polynomials might take. In 1772 Lagrange [25] expanded upon this program, highlighting the role of permutations. This led to great efforts seeking general formulas, but produced no progress. By 1801 Gauss [17] had conjectured that no such formulas existed. Indeed, in 1799 Ruffini [28] claimed to have proven this for the quintic, but his proof had a gap. This gap was closed in 1824 by Abel [1, 2]. In 1830 Galois [14, 15] developed a general theory about when such formulas exist, but his main result did not appear until 1846. In 1858 Hermite [23] gave a formula for the roots of the general quintic that involved transcendental functions.

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