Tenth Homework: MATH 410
Due Monday, 4 November 2019

1. Show that $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = \sin(1/x)$ is not uniformly continuous.
2. Second exercise on page 44 of the notes.
3. Third exercise on page 44 of the notes.
4. Prove Proposition 8.6 on page 46 in the notes.
5. Give a counterexample to each of the following false assertions.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing over $\mathbb{R}$ then $f' > 0$ over $\mathbb{R}$.
   (b) If $f : (a, b) \to \mathbb{R}$ is continuous then f has a minimum or a maximum over (a, b).
6. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be uniformly continuous over D. Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence contained in D. Show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a convergent sequence.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove it is continuous.
8. Let $f(x) = \sinh(x)$ for every $x \in \mathbb{R}$. Show that
$$\sinh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\, x^{2k+1} \quad \text{for every } x \in \mathbb{R}\,.$$
9. Evaluate the following limit. Give your reasoning.
$$\lim_{x\to 3} \frac{x^4 - 81}{x^2 - 9}\,.$$
10. Suppose that $f : (a, b) \to \mathbb{R}$ is twice differentiable and that $f'' : (a, b) \to \mathbb{R}$ is bounded over (a, b). Show that there exists an $M \in \mathbb{R}_+$ such that for every $x, y \in (a, b)$ we have
$$\big|f'(x) - f'(y)\big| \leq M\,|x - y|\,.$$
11. Prove that for every $x > 0$ we have
$$1 + \tfrac{3}{2}x < (1 + x)^{\frac{3}{2}} < 1 + \tfrac{3}{2}x + \tfrac{3}{8}x^2\,.$$
12. Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0, 1]$ if there exists a $C \in \mathbb{R}_+$ such that for every $x, y \in D$ we have
$$|f(x) - f(y)| \leq C\,|x - y|^{\alpha}\,.$$
Show that if $f : D \to \mathbb{R}$ is Hölder continuous of order α for some $\alpha \in (0, 1]$ then it is uniformly continuous over D.
13. Let $\alpha \in (0, 1)$. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^{\alpha}$. Show that f is uniformly continuous over $[0, \infty)$. Hint: Use the previous problem after showing that
$$\big|x^{\alpha} - y^{\alpha}\big| \leq |x - y|^{\alpha} \quad \text{for every } x, y \in [0, \infty)\,.$$
14. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the equation $f'(x) = 0$ has at most one solution over $x \in \mathbb{R}$. Show the equation $f(x) = 0$ has at most two solutions over $x \in \mathbb{R}$.
15. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Write negations of the following assertions.
   (a) "For all sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ contained in D we have
$$\lim_{k\to\infty} |x_k - y_k| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \big|f(x_k) - f(y_k)\big| = 0\,."$$
   (b) "For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all points $x, y \in D$ we have
$$|x - y| < \delta \quad \Longrightarrow \quad \big|f(x) - f(y)\big| < \epsilon\,."$$