Fourth Homework: MATH 410
Due Wednesday, 25 September 2019

1. Let $\{a_k\}$ be a nonincreasing, positive sequence. Prove that
$$\sum_{k=1}^{\infty} a_k \quad \text{converges} \quad \Longleftrightarrow \quad \sum_{k=0}^{\infty} 5^k a_{5^k} \quad \text{converges}\,.$$
2. Complete the proof of Proposition 3.10 in the notes by showing that
$$\lim_{k\to\infty} s_{2k} = s \quad \text{and} \quad \lim_{k\to\infty} s_{2k+1} = s \qquad \Longrightarrow \qquad \lim_{k\to\infty} s_k = s\,.$$
3. Prove Proposition 3.12 in the notes.
4. Let $\{a_k\}_{k\in\mathbb{N}}$ be a real sequence and $\{a_{n_k}\}$ be any subsequence. Show that
$$\sum_{k=0}^{\infty} a_k \quad \text{converges absolutely} \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} a_{n_k} \quad \text{converges absolutely}\,.$$
5. Give examples of both a divergent series and a convergent series such that
$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = 1\,.$$
6. Consider the set
$$\left\{ x \in \mathbb{R} \,:\, \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!}\,\frac{n!}{(3n)!}\, x^n \quad \text{converges} \right\}.$$
Use the root test to prove that this set is an interval and find its endpoints. You may use the fact that
$$\lim_{k\to\infty} \frac{\sqrt[k]{k!}}{k} = \frac{1}{e}\,.$$
7. Prove the divergence assertion of Proposition 3.15 in the notes. Show that if neither condition of Proposition 3.15 is satisfied then the series may either converge or diverge.
8. Prove Proposition 3.17 in the notes.
9. Prove Proposition 3.18 in the notes.
10. Consider the set
$$\left\{ x \in \mathbb{R} \,:\, \sum_{n=0}^{\infty} \frac{(4n)!}{(2n)!}\,\frac{n!}{(3n)!}\, x^n \quad \text{converges} \right\}.$$
Use the ratio test to prove that this set is an interval and find its endpoints.
11. Determine all $x, p \in \mathbb{R}$ for which the Fourier p-series
$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^p} \quad \text{converges}\,.$$