Second In-Class Exam Solutions
Math 410, Professor David Levermore
Friday, 1 November 2019

1. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let c be a limit point of $D \cap (c, \infty)$. Write negations of the following assertions.
   (a) "For every sequence $\{x_k\}_{k\in\mathbb{N}} \subset D \cap (c, \infty)$ we have
   $$\lim_{k\to\infty} x_k - c = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} f(x_k) = -\infty \,."$$
   (b) "For every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for every $x \in D$ we have
   $$0 < x - c < \delta \quad \Longrightarrow \quad f(x) < M \,."$$

   Remark. The assertions in (a) and (b) are equivalent.

   Solution (a). There exists a sequence $\{x_k\}_{k\in\mathbb{N}} \subset D \cap (c, \infty)$ such that
   $$\lim_{k\to\infty} x_k - c = 0 \quad \text{and} \quad \limsup_{k\to\infty} f(x_k) > -\infty \,.$$

   Remark. The negation of "$\lim_{k\to\infty} f(x_k) = -\infty$" is "$\limsup_{k\to\infty} f(x_k) > -\infty$," not "$\lim_{k\to\infty} f(x_k) > -\infty$."

   Solution (b). There exists $M \in \mathbb{R}$ such that for every $\delta > 0$ there exists $x \in D$ such that
   $$0 < x - c < \delta \quad \text{and} \quad f(x) \geq M \,.$$

2. [10] Give (with reasoning) a counterexample to each of the following false assertions.
   (a) If $f : (a, b) \to \mathbb{R}$ is increasing and one-to-one then it is continuous over (a, b).
   (b) If $f : (a, b) \to \mathbb{R}$ is differentiable and decreasing then $f' < 0$ over (a, b).

   Solution (a). There are many counterexamples. A simple one is
   $$f(x) = \begin{cases} x + 1 & \text{for } x \geq 0 \,, \\ x & \text{for } x < 0 \,. \end{cases}$$
   This function is clearly increasing (and thereby one-to-one) over $\mathbb{R}$ but it is false that f is continuous over $\mathbb{R}$ because at $x = 0$ we have
   $$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0 < 1 = f(0) \,.$$

   Solution (b). There are many counterexamples. A simple one is
   $$f(x) = -x^3 \qquad \text{over } (-\infty, \infty) \,.$$
   Because f is differentiable with
   $$f'(x) = -3x^2 \qquad \text{over } (-\infty, \infty) \,.$$
   We see that $f'(0) = 0$ and that $f'(x) < 0$ for every $x \neq 0$.

   Because $f'(x) < 0$ over $(-\infty, 0)$ and over $(0, \infty)$, the Monotonicity Theorem says that f is decreasing over $(-\infty, 0]$ and over $[0, \infty)$. Because f is decreasing over $(-\infty, 0]$ and over $[0, \infty)$, it is decreasing over $\mathbb{R}$. Therefore $f : \mathbb{R} \to \mathbb{R}$ is differentiable and decreasing over $\mathbb{R}$ but it is false that $f' < 0$ because $f'(0) = 0$. □

3. [15] Let $f : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$ with $f'(c) > 0$. Show that there exists a $\delta > 0$ such that

$$x \in (c - \delta, c) \subset (a, b) \implies f(x) < f(c)\,,$$
$$x \in (c, c + \delta) \subset (a, b) \implies f(c) < f(x)\,.$$

Remark. You are being asked to prove the first part of the Transversality Lemma (Proposition 6.2).

Solution. Because f is differentiable at c we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}\,.$$

By the ϵ-δ definition of limit, this means that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in (a, b)$ we have

$$0 < |x - c| < \delta \qquad \implies \qquad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon\,.$$

Because $f'(c) > 0$ we may take $\epsilon = f'(c)$ above to conclude that there exists $\delta > 0$ such that for every $x \in (a, b)$ we have

$$0 < |x - c| < \delta \qquad \implies \qquad \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < f'(c)\,.$$

Because $c \in (a, b)$ we may assume that δ is small enough so that $(c - \delta, c + \delta) \subset (a, b)$. Then we have

$$0 < |x - c| < \delta \qquad \implies \qquad 0 < \frac{f(x) - f(c)}{x - c} < 2f'(c)\,.$$

This implies that $x - c$ and $f(x) - f(c)$ will have the same sign when $0 < |x - c| < \delta$. It follows that

$$x \in (c - \delta, c) \quad \implies \quad x - c < 0 \quad \implies \quad f(x) - f(c) < 0\,,$$
$$x \in (c, c + \delta) \quad \implies \quad x - c > 0 \quad \implies \quad f(x) - f(c) > 0\,.$$

Because $(c - \delta, c) \subset (a, b)$ and $(c, c + \delta) \subset (a, b)$, the result follows. □

4. [15] If $f(x) = \sin(x)$ for every $x \in \mathbb{R}$ then for every $k \in \mathbb{N}$ we have

$$f^{(2k)}(x) = (-1)^k \sin(x)\,, \qquad f^{(2k+1)}(x) = (-1)^k \cos(x) \qquad \text{for every } x \in \mathbb{R}\,.$$

Use this fact to show that

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1} \quad \text{for every } x \in \mathbb{R}\,.$$

Remark. Many convergence tests can be applied to show that this series converges absolutely. For example, if we apply the Ratio Test then because for every $x \in \mathbb{R}$ we have

$$\lim_{k \to \infty} \frac{\dfrac{1}{(2k + 3)!} |x|^{2k+3}}{\dfrac{1}{(2k + 1)!} |x|^{2k+1}} = \lim_{k \to \infty} \frac{|x|^2}{(2k + 2)(2k + 3)} = 0\,,$$

we conclude for every $x \in \mathbb{R}$ that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} \qquad \text{converges absolutely}\,.$$

However, such convergence tests do not show that the series converges to $\sin(x)$, which is what we are being asked to show!

Solution. Because $f(x) = \sin(x)$, for every $k \in \mathbb{N}$ we have

$$f^{(2k)}(0) = 0\,, \qquad\qquad f^{(2k+1)}(0) = (-1)^k\,.$$

We see that the series is just the formal Taylor series for $\sin(x)$ centered at 0. The n^{th} partial sum of this series can be expressed as a Taylor polynomial approximation in two ways:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} = T_0^{2n+1} \sin(x) = T_0^{2n+2} \sin(x)\,.$$

If we use the last expression then the Lagrange Remainder Theorem states that for every $x \in \mathbb{R}$ there exists some p between 0 and x such that

$$\sin(x) = T_0^{2n+2} \sin(x) + \frac{(-1)^{n+1}}{(2n+3)!} \cos(p) x^{2n+3}\,.$$

Hence, because $|\cos(p)| \leq 1$ for every $p \in \mathbb{R}$, we have the bound

$$\left| \sin(x) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1} \right| \leq \frac{1}{(2n+3)!}\, |x|^{2n+3}\,.$$

However, either because factorials grow faster than exponentials, or because the series converges, for every $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{(2n+3)!}\, |x|^{2n+3} = 0\,.$$

Therefore the sequence of partial sums converges to $\sin(x)$. □

5. [15] Let $\alpha \in (0,1)$ and $x > 0$. Define $g : [x,\infty) \to \mathbb{R}$ by

$$g(y) = (y-x)^\alpha - y^\alpha + x^\alpha \qquad \text{for every } y \in [x,\infty)\,.$$

Prove that g is increasing over $[x,\infty)$.

Solution. The function g is continuous over $[x,\infty)$ and is continuously differentiable over (x,∞) with

$$g'(y) = \alpha(y-x)^{\alpha-1} - \alpha y^{\alpha-1} \qquad \text{for every } y \in (x,\infty)\,.$$

Because $\alpha < 1$ the function $r \mapsto r^{\alpha-1}$ is decreasing over $r > 0$, whereby

$$0 < y - x < y \quad \Longrightarrow \quad (y-x)^{\alpha-1} > y^{\alpha-1}\,.$$

Hence, because $\alpha \in (0,1)$, we have

$$g'(y) = \alpha\big[(y-x)^{\alpha-1} - y^{\alpha-1}\big] > 0 \qquad \text{for every } y \in (x,\infty)\,.$$

Therefore the Monotonicity Theorem implies that g is increasing over $[x,\infty)$. □

Remark. This is the key step in a proof of the homework problem that asked you to show for every $\alpha \in (0, 1]$ that

$$|y^\alpha - x^\alpha| \leq |y - x|^\alpha \qquad \text{for every } x, y \in [0, \infty)\,.$$

6. [15] Let $p > 1$. Prove that

$$1 + p\,x \leq (1 + x)^p \qquad \text{for every } x \geq -1\,.$$

Solution. One approach to this problem uses the Monotonicity Theorem. Define $g(x) = (1 + x)^p - 1 - p\,x$ for every $x \geq -1$. Then g is continuous over $[-1, \infty)$ and is continuously differentiable over $(-1, \infty)$ with

$$g'(x) = p\big[(1 + x)^{p-1} - 1\big]\,.$$

Clearly, $g'(x) < 0$ for $x \in (-1, 0)$ while $g'(x) > 0$ for $x > 0$. By the Monotonicity Theorem, g is decreasing over $[-1, 0]$ and g is increasing over $[0, \infty)$. Therefore the global minimum of g over $\mathbb{R}$ is $g(0) = 0$. Hence, for every $x \in [-1, \infty)$ we have

$$(1 + x)^p - 1 - p\,x = g(x) \geq g(0) = 0\,.$$

The result follows. □

Second Solution. Another approach to this problem uses convexity ideas. Define $f(x) = (1 + x)^p$ for every $x \in [-1, \infty)$. Then f is continuously differentiable over $[-1, \infty)$ with

$$f'(x) = p(1 + x)^{p-1}\,,$$

and is twice continuously differentiable over $(-1, \infty)$ with

$$f''(x) = p(p - 1)(1 + x)^{p-2} > 0\,.$$

The Monotonicity Theorem applied to f' shows that f' is increasing over $[-1, \infty)$. The Convexity Characterization Theorem then implies that f is *strictly convex* over $[-1, \infty)$. This strict convexity implies that

$$f(x) - f(0) - f'(0)x > 0 \qquad \text{for every nonzero } x \in \mathbb{R}\,.$$

Therefore $(1 + x)^p - 1 - p\,x \geq 0$ for every $x \in [-1, \infty)$. The result follows. □

Third Solution. Yet another approach uses the Lagrange Remainder Theorem. Define $f(x) = (1 + x)^p$ for every $x \in [-1, \infty)$. Then f is twice differentiable over $x > (-1, \infty)$ with

$$f'(x) = p(1 + x)^{p-1}\,, \qquad f''(x) = p(p - 1)(1 + x)^{p-2}\,.$$

By the Lagrange Remainder Theorem for every $x \in [-1, \infty)$ there exists a q between 0 and x such that

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2} f''(q)x^2\,.$$

Hence, for every $x \in [0, \infty)$ we have

$$(1 + x)^p - 1 - p\,x = p(p - 1)(1 + q)^{p-2}x^2 \geq 0\,.$$

The result follows. □

7. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the equation $f'(x) = 0$ has at most two real solutions. Prove that the equation $f(x) = 0$ has at most three real solutions.

Solution. Suppose that the equation $f(x) = 0$ has (at least) four real solutions $\{x_0, x_1, x_2, x_3\}$. Without loss of generality we can assume that

$$-\infty < x_0 < x_1 < x_2 < x_3 < \infty .$$

Then for each $i = 1, 2, 3$ we know that

- $f : [x_{i-1}, x_i] \to \mathbb{R}$ is differentiable (and hence is continuous),
- $f(x_{i-1}) = f(x_i) = 0$.

The Rolle Theorem then implies that for each $i = 1, 2, 3$ there exists $p_i \in (x_{i-1}, x_i)$ such that $f'(p_i) = 0$. Because the intervals $\{(x_{i-1}, x_i)\}_{i=1}^{4}$ are disjoint, the points $\{p_i\}_{i=1}^{3}$ are distinct. Therefore equation $f'(x) = 0$ has at least three real solutions, which shows that it has more than two real solutions. □

Second Solution. Because the equation $f'(x) = 0$ has at most two real solutions, it can have either no real solution, exactly one real solution, or exactly two real solutions. We consider each of these three cases separately.

If $f'(x) = 0$ has no real solution then by the Sign Dichotomy Theorem f' must be either negative or positive over $\mathbb{R}$. The Monotonicity Theorem then implies that f must be strictly monotonic (and hence one-to-one) over $\mathbb{R}$. The equation $f(x) = 0$ can thereby have at most one real solution.

If $f'(x) = 0$ has exactly one real solution, $x = c$, then by the Sign Dichotomy Theorem f' must be either negative or positive over each of the two disjoint intervals

$$(-\infty, c) , \qquad (c, \infty) .$$

The Monotonicity Theorem then implies that f must be strictly monotonic (and hence one-to-one) over each of the two intervals

$$(-\infty, c] , \qquad [c, \infty) .$$

Because the union of these intervals is $\mathbb{R}$, the equation $f(x) = 0$ can thereby have at most two real solutions.

If $f'(x) = 0$ has exactly two real solutions, $x = c_1$ and $x = c_2$ with $c_1 < c_2$, then by the Sign Dichotomy Theorem f' must be either negative or positive over each of the three disjoint intervals

$$(-\infty, c_1) , \quad (c_1, c_2) , \quad (c_2, \infty) .$$

The Monotonicity Theorem then implies that f must be strictly monotonic (and hence one-to-one) over each of the three intervals

$$(-\infty, c_1] , \quad [c_1, c_2] , \quad [c_2, \infty) .$$

Because the union of these intervals is $\mathbb{R}$, the equation $f(x) = 0$ can thereby have at most three real solutions. □

Remark. The second solution rests upon the Sign Dichotomy Theorem and the Monotonicity Theorem. This is heavier machinery than was used in the first solution, which rests only upon Rolle's Theorem. Indeed, the Monotonicity Theorem rests upon the Mean-Value Theorem, the proof of which rests upon Rolle's Theorem.

Exercise. Modify each of the above proofs to prove the following fact. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $n \in \mathbb{Z}_+$. Suppose that the equation $f'(x) = 0$ has at most n real solutions. Show that the equation $f(x) = 0$ has at most $n + 1$ real solutions.

8. [10] Let $D \subset \mathbb{R}$. Recall that a function $f : D \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0, 1)$ if there exists a $C \in \mathbb{R}_+$ such that

$$|f(y) - f(x)| \leq C\,|y - x|^\alpha \qquad \text{for every } x, y \in D\,.$$

Prove that if $f : D \to \mathbb{R}$ is Hölder continuous of order α for some $\alpha \in (0, 1)$ then it is uniformly continuous over D.

Solution. Let $\epsilon > 0$. Pick $\delta = (\epsilon/C)^{\frac{1}{\alpha}}$. Then, because for any $\alpha > 0$ the function $r \mapsto r^\alpha$ is increasing over $[0, \infty)$, for every $x, y \in D$ we have

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| \leq C\,|x - y|^\alpha < C\,\delta^\alpha = \epsilon\,.$$

Therefore $f : D \to \mathbb{R}$ is uniformly continuous over D. $\square$