Second In-Class Exam
Math 410, Professor David Levermore
Friday, 1 November 2019

1. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Let c be a limit point of $D \cap (c, \infty)$. Write negations of the following assertions.
   (a) "For every sequence $\{x_k\}_{k\in\mathbb{N}} \subset D \cap (c, \infty)$ we have
   $$\lim_{k\to\infty} x_k - c = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} f(x_k) = -\infty \,.\text{"}$$
   (b) "For every $M \in \mathbb{R}$ there exists a $\delta > 0$ such that for every $x \in D$ we have
   $$0 < x - c < \delta \quad \Longrightarrow \quad f(x) < M \,.\text{"}$$

2. [10] Give (with reasoning) a counterexample to each of the following false assertions.
   (a) If $f : (a, b) \to \mathbb{R}$ is increasing and one-to-one then it is continuous over (a, b).
   (b) If $f : (a, b) \to \mathbb{R}$ is differentiable and decreasing then $f' < 0$ over (a, b).

3. [15] Let $f : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$ with $f'(c) > 0$. Show that there exists a $\delta > 0$ such that
   $$x \in (c - \delta, c) \subset (a, b) \implies f(x) < f(c) \,,$$
   $$x \in (c, c + \delta) \subset (a, b) \implies f(c) < f(x) \,.$$

4. [15] If $f(x) = \sin(x)$ for every $x \in \mathbb{R}$ then for every $k \in \mathbb{N}$ we have
   $$f^{(2k)}(x) = (-1)^k \sin(x) \,, \qquad f^{(2k+1)}(x) = (-1)^k \cos(x) \qquad \text{for every } x \in \mathbb{R} \,.$$
   Use this fact to show that
   $$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \, x^{2k+1} \quad \text{for every } x \in \mathbb{R} \,.$$

5. [15] Let $\alpha \in (0, 1)$ and $x > 0$. Define $g : [x, \infty) \to \mathbb{R}$ by
   $$g(y) = (y - x)^\alpha - y^\alpha + x^\alpha \qquad \text{for every } y \in [x, \infty) \,.$$
   Prove that g is increasing over $[x, \infty)$.

6. [15] Let $p > 1$. Prove that
   $$1 + p\,x \le (1 + x)^p \qquad \text{for every } x \ge -1 \,.$$

7. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the equation $f'(x) = 0$ has at most two real solutions. Prove that the equation $f(x) = 0$ has at most three real solutions.

8. [10] Let $D \subset \mathbb{R}$. Recall that a function $f : D \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0, 1)$ if there exists a $C \in \mathbb{R}_+$ such that
   $$|f(y) - f(x)| \le C\, |y - x|^\alpha \qquad \text{for every } x, y \in D \,.$$
   Prove that if $f : D \to \mathbb{R}$ is Hölder continuous of order α for some $\alpha \in (0, 1)$ then it is uniformly continuous over D.