First In-Class Exam Solutions
Math 410, Professor David Levermore
Friday, 4 October 2019

No books, notes, calculators, or any electronic devices. Indicate your answer to each part of each question clearly. Work that you do not want considered should be crossed out. **Your reasoning must be given for full credit.** Good luck!

1. [10] Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$.
   Write the negations of the following assertions.

   (a) [5] "For some $\epsilon > 0$ we have $|b_j - 5| \geq \epsilon$ frequently as $j \to \infty$."

   (b) [5] "Every sequence in A has a subsequence that converges to a limit in A."

   Solution (a). "For every $\epsilon > 0$ we have $|b_j - 5| < \epsilon$ eventually as $j \to \infty$."

   Solution (b). "There exists a sequence in A such that every subsequence of it either diverges or converges to a limit outside A." □

   Remark. The answer "There exists a sequence in A such that no subsequence of it converges to a limit in A." does not fully carry the negation through.

   Remark. Assertion (a) is that the sequence $\{b_k\}$ does not converge to 5. Assertion (b) is the definition that the set A is sequentially compact.

2. [15] Give a counterexample to each of the following false assertions.

   (a) [5] If a real sequence $\{b_k\}_{k\in\mathbb{N}}$ diverges then the subsequence $\{b_{2k}\}_{k\in\mathbb{N}}$ diverges.

   (b) [5] If $\lim_{k\to\infty} a_k = 0$ then $\sum_{k=0}^{\infty} a_k$ converges.

   (c) [5] A countable union of closed subsets of $\mathbb{R}$ is closed.

   Solution (a). A simple counterexample is $b_k = (-1)^k$. Clearly $\{b_k\}_{k\in\mathbb{N}}$ diverges but $\{b_{2k}\}_{k\in\mathbb{N}} = \{1\}_{k\in\mathbb{N}}$ converges.

   Solution (b). The harmonic series or any p-series for some $p \in (0, 1]$ are simple counterexamples. For example,

$$\sum_{k=1}^{\infty} \frac{1}{k}, \quad \text{or} \quad \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}.$$

   Solution (c). A simple counterexample is the countable collection of closed intervals given by $[0, 1 - 2^{-n}]$ for every $n \in \mathbb{N}$. Each $[0, 1 - 2^{-n}]$ is closed but their union

$$\bigcup_{n\in\mathbb{N}} [0, 1 - 2^{-n}] = [0, 1) \quad \text{is not closed}.$$

3. [10] Let $\{c_k\}_{k\in\mathbb{N}}$ be a real sequence that diverges to ∞ as $k\to\infty$. Show that every subsequence $\{c_{n_k}\}_{k\in\mathbb{N}}$ of $\{c_k\}_{k\in\mathbb{N}}$ also diverges to ∞ as $k\to\infty$.

Remark. To show that $\{c_k\}_{k\in\mathbb{N}}$ diverges to ∞ as $k\to\infty$, we must show for every $b\in\mathbb{R}$ that $c_{n_k} > b$ eventually.

Solution. Let $b\in\mathbb{R}$. Because $\{c_k\}_{k\in\mathbb{N}}$ diverges to ∞ as $k\to\infty$, there exists $m\in\mathbb{N}$ such that

$$k \geq m \quad \Longrightarrow \quad c_k > b\,.$$

Because $n_k \geq k$ for every $k\in\mathbb{N}$, we see that

$$k \geq m \quad \Longrightarrow \quad n_k \geq m \quad \Longrightarrow \quad c_{n_k} > b\,.$$

Hence, $c_{n_k} > b$ eventually. Therefore $\{c_k\}_{k\in\mathbb{N}}$ diverges to ∞ as $k\to\infty$. □

4. [15] Let $a_0 > 0$ and define $\{a_k\}_{k\in\mathbb{N}}$ by $a_{k+1} = \frac{1}{2}(a_k + 3/a_k)$ for every $k\in\mathbb{N}$.
   (a) [10] Prove that $\{a_k\}_{k\in\mathbb{N}}$ converges.
   (b) [5] Evaluate $\lim_{k\to\infty} a_k$.

Remark. We will show that $\{a_k\}_{k\in\mathbb{N}}$ is contracting, whereby it will converge.

Solution (a). First, from the fact $a_0 > 0$ and the recursion relation

$$a_{k+1} = \frac{1}{2}\left(a_k + \frac{3}{a_k}\right) \qquad \text{for every } k\in\mathbb{N}\,,$$

it is evident by induction that

$$a_k > 0 \qquad \text{for every } k\in\mathbb{N}\,.$$

Next, upon squaring the recursion relation we see that

$$\begin{aligned} a_{k+1}^2 &= \frac{1}{4}\left(a_k + \frac{3}{a_k}\right)^2 = \frac{1}{4}\left(a_k^2 + 6 + \frac{9}{a_k^2}\right) = \frac{1}{4}\left(12 + a_k^2 - 6 + \frac{9}{a_k^2}\right) \\ &= \frac{1}{4}\left(12 + \left(a_k - \frac{3}{a_k}\right)^2\right) \geq 3 \qquad \text{for every } k\in\mathbb{N}\,. \end{aligned}$$

This shows that

$$a_k \geq \sqrt{3} \qquad \text{for every } k \geq 1\,.$$

From the recursion relation we obtain

$$\begin{aligned} a_{k+1} - a_k &= \frac{1}{2}\left(a_k + \frac{3}{a_k}\right) - \frac{1}{2}\left(a_{k-1} + \frac{3}{a_{k-1}}\right) \\ &= \frac{1}{2}\left(a_k - a_{k-1}\right)\left(1 - \frac{3}{a_{k-1}a_k}\right) \qquad \text{for every } k \geq 1\,. \end{aligned}$$

Because $a_k \geq \sqrt{3}$ for every $k \geq 1$, we see that

$$a_{k-1}a_k \geq 3 \qquad \text{for every } k \geq 2\,,$$

whereby

$$|a_{k+1} - a_k| \leq \tfrac{1}{2}|a_k - a_{k-1}| \qquad \text{for every } k \geq 2\,.$$

Therefore $\{a_k\}_{k\in\mathbb{N}}$ is a contracting sequence, which thereby converges. □

Alternaitve Solution (a). As was done in the solution above, first show that

$$a_k \geq \sqrt{3} \qquad \text{for every } k \geq 1\,.$$

Then the recursion relation implies that

$$a_{k+1} - a_k = \frac{1}{2}\left(a_k + \frac{3}{a_k}\right) - a_k = \frac{1}{2}\left(\frac{3}{a_k} - a_k\right) = \frac{3 - a_k^2}{2a_k} \leq 0 \qquad \text{for every } k \geq 1\,.$$

Therefore $\{a_k\}_{k\in\mathbb{Z}_+}$ is a nonincreasing sequence that is bounded below, whereby it converges by the Monotonic Sequence Theorem. □

Solution (b). By Part (a) we know that for some $a \geq 0$ we have

$$\lim_{k\to\infty} a_k = a\,.$$

Moreover, because $a_k \geq \sqrt{3}$ for every $k \geq 1$, we know that

$$a = \lim_{k\to\infty} a_k \geq \sqrt{3}\,.$$

By passing to the limit in the recursion relation we find that a satisfies

$$a = \lim_{k\to\infty} a_{k+1} = \frac{1}{2}\lim_{k\to\infty}\left(a_k + \frac{3}{a_k}\right) = \frac{1}{2}\left(a + \frac{3}{a}\right)\,.$$

This simplifies to $a^2 = 3$. Therefore

$$\lim_{k\to\infty} a_k = \sqrt{3}\,.$$

□

Remark. The sequence $\{a_k\}_{k\in\mathbb{N}}$ is the sequence of Newton approximations to $\sqrt{3}$. We will return to them later in the course.

5. [10] Let A and B be any subsets of $\mathbb{R}$. Prove that $A^c \cup B^c \subset (A \cup B)^c$.
(Here S^c denotes the closure of any $S \subset \mathbb{R}$.)

Remark. One proof can be built around the fact that if $C \subset D$ then their closures satisfy $C^c \subset D^c$. (This fact follows directly from the definition of closure.)

Solution. Because $A \subset (A \cup B)$ and $B \subset (A \cup B)$, we know $A^c \subset (A \cup B)^c$ and $B^c \subset (A \cup B)^c$. We conclude that $A^c \cup B^c \subset (A \cup B)^c$. □

Remark. Another proof shows directly from the definition of closure that every element of $A^c \cup B^c$ is also an element of $(A \cup B)^c$.

Alternate Solution. Let $x \in A^c \cup B^c$ be arbitrary. Then either $x \in A^c$ or $x \in B^c$. (Both can be true.) Without loss of generality we can assume that $x \in A^c$. Then by the definition of closure there exists a sequence $\{x_n\} \subset A$ such that $x_n \to x$ as $n \to \infty$. Because $A \subset (A \cup B)$ we have $\{x_n\} \subset A \cup B$ and $x_n \to x$ as $n \to \infty$. By the definition of closure, we have $x \in (A \cup B)^c$. But because $x \in A^c \cup B^c$ was arbitrary, we conclude that $A^c \cup B^c \subset (A \cup B)^c$. □

6. [10] Let $\{a_k\}$ be a nondecreasing sequence in $\mathbb{R}$. Show that it converges if it has a convergent subsequence.

Remark. Our proof will use the Monotonic Sequence Theorem, which says that a nondecreasing real sequence converges if and only if it is bounded above.

Solution. Let $\{a_{n_k}\}$ be a convergent subsequence of $\{a_k\}$. Because any subsequence of a nondecreasing sequence is also nondecreasing, $\{a_{n_k}\}$ is nondeceasing. Because $\{a_{n_k}\}$ is a nondeceasing sequence that converges, the Monotonic Sequence Theorem implies that it is bounded above. Let $M \in \mathbb{R}$ be an upper bound of $\{a_{n_k}\}$, so that

$$a_{n_k} \leq M \quad \text{for every } k\,.$$

Because $k \geq n_k$ for every k and $\{a_k\}$ is nondecreasing, we have

$$a_k \leq a_{n_k} \quad \text{for every } k\,.$$

By putting the above two inequalities together, we see that

$$a_k \leq a_{n_k} \leq M \quad \text{for every } k\,.$$

Therefore the nondecreasing sequence $\{a_k\}$ is also bounded above by M. The Monotonic Sequence Theorem thereby implies that it converges. □

Remark. In fact, a monotonic sequence converges if and only if it has a convergent subsequence.

7. [10] Let $\{b_k\}$ be a nonzero real sequence. Prove that

$$\liminf_{k\to\infty} \frac{\log(|b_k|)}{\log(\frac{1}{k})} > 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} b_k \quad \text{converges absolutely}\,.$$

(This is the convergence conclusion of the Log Test.)

Solution. Because

$$\liminf_{k\to\infty} \frac{\log(|b_k|)}{\log(\frac{1}{k})} > 1\,,$$

there exists $p > 1$ such that

$$\liminf_{k\to\infty} \frac{\log(|b_k|)}{\log(\frac{1}{k})} > p\,.$$

By Proposition 2.17 we have

$$\frac{\log(|b_k|)}{\log(\frac{1}{k})} > p \quad \text{eventually}\,.$$

Because $\log(\frac{1}{k}) < 0$ eventually, we see that

$$\log(|b_k|) < p \log\left(\frac{1}{k}\right) = \log\left(\frac{1}{k^p}\right) \quad \text{eventually}\,.$$

Because log is an increasing function we see that

$$|b_k| < \frac{1}{k^p} \quad \text{eventually}\,.$$

The *Direct Comparison Test* with the p-series shows that because

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \quad \text{converges for every } p > 1\,,$$

we know that

$$\sum_{k=0}^{\infty} |b_k| \quad \text{converges}\,.$$

The definition of absolute convergence then implies that

$$\sum_{k=0}^{\infty} b_k \quad \text{converges absolutely}\,.$$

□

8. [20] Determine the set of all $x \in \mathbb{R}$ for which

$$\sum_{k=2}^{\infty} \frac{2^k x^k}{\log(k)} \quad \text{converges}\,.$$

Give your reasoning. (The set is an interval. Be sure to check its endpoints!)

Solution. Let a_k denote the k^{th} term in the sum, namely let

$$a_k = \frac{2^k x^k}{\log(k)}\,.$$

We have

$$\limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \limsup_{k\to\infty} \frac{2^{k+1}|x|^{k+1}}{\log(k+1)} \frac{\log(k)}{2^k|x|^k} = 2|x| \limsup_{k\to\infty} \frac{\log(k)}{\log(k+1)}\,.$$

By the l'Hôpital Rule

$$\lim_{k\to\infty} \frac{\log(k)}{\log(k+1)} = \lim_{k\to\infty} \frac{\dfrac{1}{k}}{\dfrac{1}{k+1}} = \lim_{k\to\infty} \frac{k+1}{k} = 1\,.$$

Therefore

$$\limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 2|x| \lim_{k\to\infty} \frac{\log(k)}{\log(k+1)} = 2|x|\,.$$

The *Ratio Test* then concludes that the series *converges absolutely* when $2|x| < 1$ and *diverges* when $2|x| > 1$. The *Ratio Test* says nothing when $2|x| = 1$.

When $2x = -1$ the series becomes

$$\sum_{k=2}^{\infty} (-1)^k \frac{1}{\log(k)}\,.$$

Because the terms $1/\log(k)$ are positive and decreasing with

$$\lim_{k\to\infty} \frac{1}{\log(k)} = 0\,,$$

the *Alternating Series Test* can be applied to show that the series *converges*.

When $2x = 1$ the series becomes

$$\sum_{k=2}^{\infty} \frac{1}{\log(k)} .$$

Because $\log(x) \leq (x-1)$ for every $x \in \mathbb{R}_+$, we see that

$$\frac{1}{k-1} \leq \frac{1}{\log(k)} \quad \text{for every } k \in \{2, 3, \cdots\} .$$

Then because the *harmonic series*

$$\sum_{k=2}^{\infty} \frac{1}{k-1} \quad \text{diverges} .$$

the *Direct Comparison Test* shows that the series *diverges*. Alternatively, because the terms $1/\log(k)$ are positive and decreasing, the *Cauchy* 2^k *Test* can be applied to show that the series *diverges*.

Therefore the set of all $x \in \mathbb{R}$ for which the series converges is the interval

$$\big[-\tfrac{1}{2}, \tfrac{1}{2}\big) .$$

□

Remark. It is not enough to argue that the series converges in the interval $[-\frac{1}{2}, \frac{1}{2})$. You also have to argue that it diverges outside the interval.

Remark. Rather than using the l'Hôpital Rule above, we could have argued that

$$\log(k+1) = \log(k) + \log\left(1 + \frac{1}{k}\right) < \log(k) + \frac{1}{k} ,$$

whereby

$$\frac{1}{1 + \dfrac{1}{k \log(k)}} < \frac{\log(k)}{\log(k+1)} < 1 .$$

It follows that

$$\lim_{k \to \infty} \frac{\log(k)}{\log(k+1)} = 1 .$$