First In-Class Exam
Math 410, Professor David Levermore
Friday, 4 October 2019

No books, notes, calculators, or any electronic devices. Indicate your answer to each part of each question clearly. Work that you do not want considered should be crossed out. **Your reasoning must be given for full credit.** Good luck!

1. [10] Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$.
   Write the negations of the following assertions.
   (a) [5] "For some $\epsilon > 0$ we have $|b_j - 5| \geq \epsilon$ frequently as $j \to \infty$."
   (b) [5] "Every sequence in A has a subsequence that converges to a limit in A."

2. [15] Give a counterexample to each of the following false assertions.
   (a) [5] If a real sequence $\{b_k\}_{k\in\mathbb{N}}$ diverges then the subsequence $\{b_{2k}\}_{k\in\mathbb{N}}$ diverges.
   (b) [5] If $\lim_{k\to\infty} a_k = 0$ then $\sum_{k=0}^{\infty} a_k$ converges.
   (c) [5] A countable union of closed subsets of $\mathbb{R}$ is closed.

3. [10] Let $\{c_k\}_{k\in\mathbb{N}}$ be a real sequence that diverges to ∞ as $k \to \infty$. Show that every subsequence $\{c_{n_k}\}_{k\in\mathbb{N}}$ of $\{c_k\}_{k\in\mathbb{N}}$ also diverges to ∞ as $k \to \infty$.

4. [15] Let $a_0 > 0$ and define $\{a_k\}_{k\in\mathbb{N}}$ by $a_{k+1} = \frac{1}{2}(a_k + 3/a_k)$ for every $k \in \mathbb{N}$.
   (a) [10] Prove that $\{a_k\}_{k\in\mathbb{N}}$ converges.
   (b) [5] Evaluate $\lim_{k\to\infty} a_k$.

5. [10] Let A and B be any subsets of $\mathbb{R}$. Prove that $A^c \cup B^c \subset (A \cup B)^c$.
   (Here S^c denotes the closure of any $S \subset \mathbb{R}$.)

6. [10] Let $\{a_k\}$ be a nondecreasing sequence in $\mathbb{R}$. Show that it converges if it has a convergent subsequence.

7. [10] Let $\{b_k\}$ be a nonzero real sequence. Prove that
   $$\liminf_{k\to\infty} \frac{\log(|b_k|)}{\log(\frac{1}{k})} > 1 \quad \Longrightarrow \quad \sum_{k=0}^{\infty} b_k \quad \text{converges absolutely}\,.$$
   (This is the convergence conclusion of the Log Test.)

8. [20] Determine the set of all $x \in \mathbb{R}$ for which
   $$\sum_{k=2}^{\infty} \frac{2^k x^k}{\log(k)} \quad \text{converges}\,.$$
   Give your reasoning. (The set is an interval. Be sure to check its endpoints!)