Quiz 11 Solutions, Math 246, Professor David Levermore
Tuesday, 3 December 2019

(1) [5] Consider the system

$$x' = -4x + y\,, \qquad y' = 5x - 5x^2\,.$$

Its stationary points are $(0,0)$ and $(1,4)$. Classify the type and stability of each of these stationary points. (You do not have to sketch anything.)

Solution. The Jacobian matrix is $\boldsymbol{\partial}\mathbf{f}(x,y) = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 5 - 10x & 0 \end{pmatrix}$.

- At $(0,0)$ the coefficient matrix of its linearization is $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(0,0) = \begin{pmatrix} -4 & 1 \\ 5 & 0 \end{pmatrix}$, which has characterisitc polynomial
$$p_{\mathbf{A}}(\zeta) = \zeta^2 + 4\zeta - 5 = (\zeta+5)(\zeta-1)\,.$$
The eigenvalues of $\mathbf{A}$ are -5 and 1. Because these are real with opposite sign, the stationary point $(0,0)$ is a *saddle* and thereby is *unstable*, but not repelling.
- At $(1,4)$ the coefficient matrix of its linearization is $\mathbf{B} = \boldsymbol{\partial}\mathbf{f}(1,4) = \begin{pmatrix} -4 & 1 \\ -5 & 0 \end{pmatrix}$, which has characteristic polynomial
$$p_{\mathbf{B}}(\zeta) = \zeta^2 + 4\zeta + 5 = (\zeta+2)^2 + 1^2\,.$$
The eigenvalues of $\mathbf{B}$ are $-2 \pm i$. Because these eigenvalues are a conjugate pair with negative real part, and because $b_{21} = -5 < 0$, the stationary point $(0,0)$ is a *clockwise spiral sink* and thereby is *attracting*.

(2) [5] Consider the planar system

$$u' = -2u + v\,, \qquad v' = -3v + 3u^2\,.$$

Its stationary points are $(0,0)$ and $(2,4)$. Classify the type and stability of each of these stationary points. (You do not have to sketch anything.)

Solution. The Jacobian matrix is $\boldsymbol{\partial}\mathbf{f}(u,v) = \begin{pmatrix} \partial_u f & \partial_v f \\ \partial_u g & \partial_v g \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 6u & -3 \end{pmatrix}$.

- At $(0,0)$ the coefficient matrix of its linearization is $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(0,0) = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}$. Because this is a triangular matrix, we can read off from its diagonal that the eigenvalues of $\mathbf{A}$ are -2 and -3. Because these are both negative, the stationary point $(0,0)$ is a *nodal sink* and thereby is *attracting*.
- At $(2,4)$ the coefficient matrix of its linearization is $\mathbf{B} = \boldsymbol{\partial}\mathbf{f}(2,4) = \begin{pmatrix} -2 & 1 \\ 12 & -3 \end{pmatrix}$, which has characteristic polynomial
$$p_{\mathbf{B}}(\zeta) = \zeta^2 + 5\zeta - 6 = (\zeta-1)(\zeta+6)\,.$$
The eigenvalues of $\mathbf{B}$ are 1 and -6. Because these are real with opposite sign, the stationary point $(2,4)$ is a *saddle* and thereby is *unstable*, but not repelling.