Quiz 10 Solutions, Math 246, Professor David Levermore
Tuesday, 26 November 2019

(1) [3] Consider the system $\mathbf{x}' = \mathbf{B}\mathbf{x}$ where $\mathbf{B} = \begin{pmatrix} -4 & 6 \\ -3 & 2 \end{pmatrix}$.

(a) [1] Classify its phase-plane portrait.
(b) [1] Determine the stability of the origin for this system.
(c) [1] Sketch its phase-plane portrait.

Solution (a). The characteristic polynomial of $\mathbf{B}$ is

$$\begin{aligned} p(z) = z^2 - \operatorname{tr}(\mathbf{B})z + \det(\mathbf{B}) &= z^2 - (-2)z + \big(-4 \cdot 2 - (-3) \cdot 6 \big) \\ &= z^2 + 2z + 10 = (z+1)^2 + 3^2 \,. \end{aligned}$$

Because this has the conjugate pair of roots $-1 \pm i3$, the phase-plane portrait of the system $\mathbf{x}' = \mathbf{B}\mathbf{x}$ is a spiral sink. Because the b_{21} entry is negative, the phase-plane portrait is a *clockwise spiral sink*.

Solution (b). The origin is *attracting* for a spiral sink.

Solution (c). Your sketch should show a curve that spirals towards the origin in a clockwise fashion.

(2) [3] Consider the system $\mathbf{x}' = \mathbf{C}\mathbf{x}$ where the 2×2 matrix $\mathbf{C}$ has eigenpairs

$$\left(1\,, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) , \qquad \left(2\,, \begin{pmatrix} 3 \\ -1 \end{pmatrix}\right) .$$

(a) [1] Classify its phase-plane portrait.
(b) [1] Determine the stability of the origin for this system.
(c) [1] Sketch its phase-plane portrait.

Solution (a). Because $\mathbf{C}$ has two positive eigenvalues, the phase-plane portrait of the system $\mathbf{x}' = \mathbf{C}\mathbf{x}$ is a *nodal source*.

Solution (b). The origin is *repelling* for a nodal source.

Solution (c). Your sketch should show the line through the origin and the point $(1, 3)$, and the line through the origin and the point $(3, -1)$. Each line should have arrows on it pointing away from the origin. These two lines separate the plane into four regions. Your sketch should show at least one representative orbit in each of these four regions. Each representative orbit should emerge from the origin tangent to the line through the point $(1, 3)$ and should bend to become more parallel to the line through the point $(3, -1)$ further away from the origin. Each representative orbit should have arrows on it pointing away from the origin.

(3) [4] Consider the system

$$x' = 3x - y\,, \qquad y' = 5x - 3y + 2x^2\,.$$

(a) [2] This system has two stationary points. Find them.
(b) [2] Find a nonconstant function $H(x,y)$ such that every orbit of this system satisfies $H(x,y) = c$ for some constant c.

Solution (a). The stationary points satisfy

$$0 = 3x - y\,, \qquad 0 = 5x - 3y + 2x^2\,.$$

The first equation is satisfied if and only if $y = 3x$, whereby the second equation becomes $0 = -4x + 2x^2 = -2x(2-x)$, which has solutions $x = 0$ and $x = 2$. Therefore the stationary points are $(0,0)$ and $(2,6)$.

Solution (b). The system is *Hamiltonian* because

$$\partial_x(3x - y) + \partial_y\big(5x - 3y + 2x^2\big) = 3 - 3 = 0\,,$$

where by the orbit equation is *exact*. Therefore there exists $H(x,y)$ such that

$$\partial_y H(x,y) = 3x - y\,, \qquad -\partial_x H(x,y) = 5x - 3y + 2x^2\,.$$

By integrating the first equation we find that

$$H(x,y) = 3xy - \tfrac{1}{2}y^2 + h(x)\,.$$

By substituting this into the second equation we see that

$$-3y - h'(x) = 5x - 3y + 2x^2\,,$$

whereby $h'(x) = -5x - 2x^2$. Therefore we can set

$$H(x,y) = 3xy - \tfrac{1}{2}y^2 - \tfrac{5}{2}x^2 - \tfrac{2}{3}x^3\,.$$

Remark. The stationary points of the system are critical points of the Hamiltonian $H(x,y)$ — i.e. points where $\partial_x H(x,y) = 0$ and $\partial_y H(x,y) = 0$. The nature of these critical points can be studied with the Hessian matrix of $H(x,y)$, which is

$$\boldsymbol{\partial}^2 H(x,y) = \begin{pmatrix} \partial_{xx}H(x,y) & \partial_{xy}H(x,y) \\ \partial_{yx}H(x,y) & \partial_{yy}H(x,y) \end{pmatrix} = \begin{pmatrix} -5-4x & 3 \\ 3 & -1 \end{pmatrix}.$$

- At the critical point $(0,0)$ we see that

  $$\det\big(\boldsymbol{\partial}^2 H(0,0)\big) = \det\begin{pmatrix} -5 & 3 \\ 3 & -1 \end{pmatrix} = -4 < 0\,,$$

  whereby the critical point $(0,0)$ is a saddle point of $H(x,y)$.
- At the critical point $(2,6)$ we see that

  $$\det\big(\boldsymbol{\partial}^2 H(2,6)\big) = \det\begin{pmatrix} -13 & 3 \\ 3 & -1 \end{pmatrix} = 4 > 0\,,$$

  $$\operatorname{tr}\big(\boldsymbol{\partial}^2 H(2,6)\big) = \operatorname{tr}\begin{pmatrix} -13 & 3 \\ 3 & -1 \end{pmatrix} = -14 < 0\,,$$

  whereby the critical point $(2,6)$ is a local maximizer of $H(x,y)$.

Therefore, with the sign conventions that we have adopted, we see that:

- the point $(0,0)$ is a saddle point in the phase-plane portrait;
- the point $(2,6)$ is a counterclockwise center in the phase-plane portrait.

Remark. The phase-plane portrait can be filled out by plotting some level sets of the Hamiltonian $H(x,y)$. The most important level set to sketch is that associated with the saddle point $(0,0)$ is the set of all points (x,y) that satisfy $H(x,y) = H(0,0) = 0$, — i.e. that satisfy

$$3xy - \tfrac{1}{2}y^2 - \tfrac{5}{2}x^2 - \tfrac{2}{3}x^3 = 0\,.$$

Because this is quadratic in y, we may solve for y as a function of x. For example, after multiplying by -2 and completing the square this becomes

$$(y - 3x)^2 - 4x^2 + \tfrac{4}{3}x^3 = 0\,,$$

whereby

$$y = 3x \pm 2x\sqrt{1 - \tfrac{1}{3}x} \qquad \text{for } x \leq 3\,.$$

These two curves may be sketched in the phase-plane using techniques from calculus. They intersect where $x = 0$ and where $x = 3$, which is at the points $(0,0)$ and $(3,9)$. Sketching other level sets is slightly harder. Alternatively, the MATLAB command "contour" can be used to plot several level sets. Because $H(2,6) = \frac{8}{3}$, a fairly good idea of the phase-plane portrait can be obtained by plotting the five level sets

$$H(x,y) = 2\,, \quad H(x,y) = 1\,, \quad H(x,y) = 0\,, \quad H(x,y) = -1\,, \quad H(x,y) = -2\,.$$

The orbits around $(2,6)$ must be counterclockwise. From that fact and the fact that $(0,0)$ is a saddle, the direction of all the orbits can be figured out. Try it. The orbits on the level set $H(x,y) = 0$ are separatrices, which separate the different behavior seen for orbits with $H(x,y) < 0$ and that seen for orbits with $H(x,y) > 0$.