**Quiz 9 Solutions, Math 246, Professor David Levermore**
**Tuesday, 12 November 2019**

(1) [5] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ -t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} 2t^4 \\ 4 - t^6 \end{pmatrix}$.

(a) [2] Compute their Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.

**Solution (a).** The Wronskian is

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 1 & 2t^4 \\ -t^2 & 4 - t^6 \end{pmatrix} = 1 \cdot (4 - t^6) - (-t^2) \cdot 2t^4 = 4 + t^6 \,.$$

**Solution (b).** Let $\mathbf{\Psi}(t) = \begin{pmatrix} 1 & 2t^4 \\ -t^2 & 4 - t^6 \end{pmatrix}$. Because $\mathbf{\Psi}'(t) = \mathbf{A}(t)\mathbf{\Psi}(t)$, we have

$$\begin{aligned} \mathbf{A}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 0 & 8t^3 \\ -2t & -6t^5 \end{pmatrix} \begin{pmatrix} 1 & 2t^4 \\ -t^2 & 4 - t^6 \end{pmatrix}^{-1} \\ &= \frac{1}{4 + t^6} \begin{pmatrix} 0 & 8t^3 \\ -2t & -6t^5 \end{pmatrix} \begin{pmatrix} 4 - t^6 & -2t^4 \\ t^2 & 1 \end{pmatrix} \\ &= \frac{1}{4 + t^6} \begin{pmatrix} 8t^5 & 8t^3 \\ -8t - 4t^7 & -2t^5 \end{pmatrix} . \end{aligned}$$

**Remark.** This problem is related to Problem 3 on Quiz 8.

(2) [4] Let $\mathbf{B} = \begin{pmatrix} -1 & -2 \\ 4 & -5 \end{pmatrix}$. Compute $e^{t\mathbf{B}}$.

**Solution.** Because $\mathbf{B}$ is $2 \times 2$ its characteristic polynomial is

$$p(\zeta) = \zeta^2 - \mathrm{tr}(\mathbf{B})\zeta + \det(\mathbf{B}) = \zeta^2 + 6\zeta + 13 = (\zeta + 3)^2 + 2^2 \,.$$

Because this is a *sum of squares* with roots $-3 \pm i2$, the $2 \times 2$ formula gives

$$\begin{aligned} e^{t\mathbf{B}} &= e^{-3t}\left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{B} + 3\mathbf{I})\right] \\ &= e^{-3t}\left[\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2}\begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}\right] \\ &= e^{-3t}\begin{pmatrix} \cos(2t) + \sin(2t) & -\sin(2t) \\ 2\sin(2t) & \cos(2t) - \sin(2t) \end{pmatrix} . \end{aligned}$$

**Remark.** An alternative solution is on the back.

(3) [1] Suppose that $e^{t\mathbf{C}} = e^{3t}\begin{pmatrix} \cosh(4t) & -\frac{1}{2}\sinh(4t) \\ -2\sinh(4t) & \cosh(4t) \end{pmatrix}$.

Compute the Green matrix $\mathbf{G}(t, s)$ for the system $\mathbf{x}' = \mathbf{C}\mathbf{x} + \mathbf{f}(t)$.

**Solution.** The Green matrix is

$$\mathbf{G}(t, s) = e^{(t-s)\mathbf{C}} = e^{3(t-s)}\begin{pmatrix} \cosh(4(t-s)) & -\frac{1}{2}\sinh(4(t-s)) \\ -2\sinh(4(t-s)) & \cosh(4(t-s)) \end{pmatrix} .$$

**Alternative Solution for 2.** Because $\mathbf{B}$ is $2 \times 2$ its characteristic polynomial is

$$p(\zeta) = \zeta^2 - \operatorname{tr}(\mathbf{B})\zeta + \det(\mathbf{B}) = \zeta^2 + 6\zeta + 13 = (\zeta+3)^2 + 2^2 .$$

This is a *sum of squares* with roots $-3 \pm i2$. Because the Green function for $p(\mathrm{D})$ satisfies the initial-value problem

$$p(\mathrm{D})g = 0 , \qquad g(0) = 0 , \quad g'(0) = 1 ,$$

it had the form

$$g(t) = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t) .$$

Because $g(0) = c_1$, the initial condition $g(0) = 0$ implies that $c_1 = 0$, whereby

$$g'(t) = c_2 e^{-3t} 2\cos(2t) - c_2 3 e^{-3t}\sin(2t) .$$

Because $g'(0) = 2c_2$, the initial condition $g'(0) = 1$ implies that $c_2 = \frac{1}{2}$, whereby

$$g(t) = \tfrac{1}{2} e^{-3t}\sin(2t) .$$

Because $p(\zeta) = \zeta^2 + 6\zeta + 13$, the natural fundamental set associated with $t_I = 0$ is

$$\begin{aligned} N_1(t) &= g(t) = \tfrac{1}{2} e^{-3t}\sin(2t) , \\ N_0(t) &= N_1'(t) + 6g(t) \\ &= e^{-3t}\cos(2t) - \tfrac{3}{2} e^{-3t}\sin(2t) + 3e^{-3t}\sin(2t) \\ &= e^{-3t}\cos(2t) + \tfrac{3}{2} e^{-3t}\sin(2t) . \end{aligned}$$

The natural fundamental set method then gives

$$\begin{aligned} e^{t\mathbf{B}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{B} \\ &= \left(e^{-3t}\cos(2t) + \tfrac{3}{2} e^{-3t}\sin(2t)\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tfrac{1}{2} e^{-3t}\sin(2t)\begin{pmatrix} -1 & -2 \\ 4 & -5 \end{pmatrix} \\ &= e^{-3t}\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tfrac{1}{2} e^{-3t}\sin(2t)\begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} \cos(2t) + \sin(2t) & -\sin(2t) \\ 2\sin(2t) & \cos(2t) - \sin(2t) \end{pmatrix} . \end{aligned}$$

**Remark.** For $2 \times 2$ matrices the $2 \times 2$ formula for the matrix exponential will always be the fastest way to go because it precomputes the natural fundamental set.