Quiz 8 Solutions, Math 246, Professor David Levermore
Tuesday, 5 November 2019

Short Table: $\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$ for $s > a$, $\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$.

(1) [5] Find $F(s) = \mathcal{L}[f](s)$ where $f(t) = u(t-4)e^{-3t} + 5\delta(t-2)$.

Solution. By linearity we have

$$\mathcal{L}[f](s) = \mathcal{L}[u(t-4)e^{-3t}](s) + 5\mathcal{L}[\delta(t-2)](s).$$

By the shifty step method $u(t-4)e^{-3t} = u(t-4)j(t-4)$ where

$$j(t) = e^{-3(t+4)} = e^{-3t}e^{-12}.$$

Hence, the second entry in our short table with $c = 4$ gives

$$\begin{aligned}\mathcal{L}[u(t-4)e^{-3t}](s) = \mathcal{L}[u(t-4)j(t-4)](s) &= e^{-4s}\mathcal{L}[j](s) \\ &= e^{-4s}e^{-12}\mathcal{L}[e^{-3t}](s).\end{aligned}$$

The first entry in our short table with $n = 0$ and $a = -3$ gives

$$\mathcal{L}[e^{-3t}](s) = \frac{1}{s+3}.$$

Therefore

$$\mathcal{L}[u(t-4)e^{-3t}](s) = e^{-4s}e^{-12}\frac{1}{s+3}.$$

By the rule for evaluating the unit impulse under an integral we have

$$\mathcal{L}[\delta(t-2)](s) = \int_0^\infty e^{-st}\delta(t-2)\,\mathrm{d}t = e^{-2s}.$$

Therefore

$$\mathcal{L}[f](s) = \frac{e^{-4s-12}}{s+3} + 5e^{-2s}.$$

(2) [2] Recast the equation $x'''' - e^{t+x''}x' + \sin(x) = 0$ as a first-order system of ordinary differential equations.

Solution. Because the equation is fourth order, the first-order system must have dimension four. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ e^{t+y_3}y_2 + \sin(y_1) \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x \\ x' \\ x'' \\ x''' \end{pmatrix}.$$

Remark. There should be no x, x', x'', or x''' appearing in the first-order system. The only place these should appear is in the dictionary on the right that shows their relationship to the new variables. The first-order system should be expressed solely in terms of the new variables. The new variables are y_1, y_2, y_3, and y_4 in the solution above. Any similar set of new variables could be used.

(3) [3] Consider the matrix-valued function $\mathbf{\Psi}(t) = \begin{pmatrix} 1 & 2t^4 \\ -t^2 & 4 - t^6 \end{pmatrix}$.

(a) Compute $\det\big(\mathbf{\Psi}(t)\big)$.
(b) Compute $\mathbf{\Psi}(t)^{-1}$.
(c) Compute $\mathbf{\Psi}'(t)$.

Solution (a). The determinant of $\mathbf{\Psi}(t)$ is

$$\det\big(\mathbf{\Psi}(t)\big) = \det\begin{pmatrix} 1 & 2t^4 \\ -t^2 & 4 - t^6 \end{pmatrix} = 1 \cdot (4 - t^6) - (-t^2) \cdot (2t^4) = 4 + t^6 \,.$$

Solution (b). The inverse of $\mathbf{\Psi}(t)$ is

$$\mathbf{\Psi}(t)^{-1} = \frac{1}{\det\big(\mathbf{\Psi}(t)\big)} \begin{pmatrix} 4 - t^6 & -2t^4 \\ t^2 & 1 \end{pmatrix} = \frac{1}{4 + t^6} \begin{pmatrix} 4 - t^6 & -2t^4 \\ t^2 & 1 \end{pmatrix} .$$

Solution (c). The derivative of $\mathbf{\Psi}(t)$ is

$$\mathbf{\Psi}'(t) = \begin{pmatrix} 0 & 8t^3 \\ -2t & -6t^5 \end{pmatrix} .$$