Quiz 7 Solutions, Math 246, Professor David Levermore
Tuesday, 29 October 2019

Short Table: $\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$ for $s > a$, $\qquad \mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$.

(1) [4] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for the function $f(t) = u(t-3)e^{-2t}$, where u is the unit step function.

Solution. By the definitions of the Laplace transform and the unit step function

$$\begin{aligned}\mathcal{L}[f](s) &= \lim_{T\to\infty}\int_0^T e^{-st} f(t)\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T e^{-st}u(t-3)e^{-2t}\,\mathrm{d}t \\ &= \lim_{T\to\infty}\int_3^T e^{-st}e^{-2t}\,\mathrm{d}t = \lim_{T\to\infty}\int_3^T e^{-(s+2)t}\,\mathrm{d}t\,.\end{aligned}$$

For $s \le -2$ we have $e^{-(s+2)t} \ge 1$, so for $T > 3$

$$\int_3^T e^{-(s+2)t}\,\mathrm{d}t \ge \int_3^T \mathrm{d}t = T - 3\,,$$

whereby $\mathcal{L}[f](s)$ is undefined for $s \le -2$ because

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\int_3^T e^{-(s+2)t}\,\mathrm{d}t \ge \lim_{T\to\infty}(T-3) = \infty \qquad \text{for } s \le -2\,.$$

For $s > -2$ and $T > 3$

$$\int_3^T e^{-(s+2)t}\,\mathrm{d}t = -\frac{e^{-(s+2)t}}{s+2}\bigg|_3^T = \frac{e^{-(s+2)3}}{s+2} - \frac{e^{-(s+2)T}}{s+2}\,,$$

whereby

$$\begin{aligned}\mathcal{L}[f](s) &= \lim_{T\to\infty}\int_3^T e^{-(s+2)t}\,\mathrm{d}t = \lim_{T\to\infty}\left[\frac{e^{-(s+2)3}}{s+2} - \frac{e^{-(s+2)T}}{s+2}\right] \\ &= \frac{e^{-(s+2)3}}{s+2} \qquad \text{for } s > -2\,.\end{aligned}$$

Remark. You must give a reason why $\mathcal{L}[f](s)$ is undefined for $s \le -2$ for full credit. You must use the definition of the Laplace transform for any credit.

Remark. While the table should not have been used in your answer, it could have been used to check your answer. By the shifty step method we see that

$$f(t) = u(t-3)e^{-2t} = u(t-3)j(t-3)\,, \qquad \text{where} \quad j(t) = e^{-2(t+3)} = e^{-2t}e^{-6}\,.$$

The second table entry with $c = 3$ and the first table entry with $n = 0$ and $a = -2$ then yield

$$\begin{aligned}\mathcal{L}[f](s) = \mathcal{L}[u(t-3)j(t-3)](s) &= e^{-3s}\mathcal{L}[j](s) = e^{-3s}e^{-6}\mathcal{L}[e^{-2t}](s) \\ &= e^{-3(s+2)}\,\frac{1}{s+2}\,.\end{aligned}$$

(2) [3] Find the Laplace transform $X(s)$ of the solution $x(t)$ of the initial-value problem

$$x'' + 16x = 0\,, \qquad x(0) = 3\,, \quad x'(0) = -5\,.$$

DO NOT solve for $x(t)$, just $X(s)$!

Solution. The Laplace transform of the differential equation is

$$\mathcal{L}[x''](s) + 16\mathcal{L}[x](s) = 0\,,$$

where the initial conditions give

$$\begin{aligned}
\mathcal{L}[x](s) &= X(s)\,,\\
\mathcal{L}[x'](s) &= s\mathcal{L}[x](s) - x(0) = sX(s) - 3\,,\\
\mathcal{L}[x''](s) &= s\mathcal{L}[x'](s) - x'(0) = s\big(sX(s) - 3\big) + 5 = s^2X(s) - 3s + 5\,.
\end{aligned}$$

By placing these into the Laplace transform of the differential equation we get

$$\big(s^2X(s) - 3s + 5\big) + 16X(s) = 0\,,$$

which yields

$$(s^2 + 16)X(s) - 3s + 5 = 0\,,$$

whereby

$$X(s) = \frac{3s - 5}{s^2 + 16}\,.$$

(3) [3] Find $y(t) = \mathcal{L}^{-1}[Y](t)$ where $Y(s) = e^{-3s}\dfrac{24}{(s-4)(s+2)}$.

Solution. By the partial fraction identity

$$J(s) = \frac{24}{(s-4)(s+2)} = \frac{4}{s-4} + \frac{-4}{s+2}\,,$$

and by the first table entry with $n = 0$ and $a = 4$ and with $n = 0$ and $a = -2$ we have

$$\begin{aligned}
j(t) = \mathcal{L}^{-1}\left[\frac{24}{(s-4)(s+2)}\right](t) &= 4\mathcal{L}^{-1}\left[\frac{1}{s-4}\right](t) - 4\mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t)\\
&= 4e^{4t} - 4e^{-2t}\,.
\end{aligned}$$

Therefore by the second table entry we have

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}[Y](t) = \mathcal{L}^{-1}\big[e^{-3s}J(s)\big](t) &= u(t-3)j(t-3)\\
&= u(t-3)\big(4e^{4(t-3)} - 4e^{-2(t-3)}\big)\\
&= 4u(t-3)\big(e^{4t-12} - e^{-2t+6}\big)\,.
\end{aligned}$$

Short Table: $\mathcal{L}[t^ne^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$ for $s > a\,,$ $\qquad \mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)\,.$