Quiz 6 Solutions, Math 246, Professor David Levermore
Tuesday, 15 October 2019

(1) [3] Find the amplitude and phase of the simple harmonic motion

$$h(t) = 5\cos(2t) - 5\sqrt{3}\sin(2t)\,.$$

Solution. The point in the plane with Cartesian coordinates $(5, -5\sqrt{3})$ lies in the fourth quadrant and has polar coordinates (a, ϕ) with

$$a = \sqrt{5^2 + 5^2 \cdot 3} = \sqrt{25 + 75} = \sqrt{100} = 10\,,$$
$$\phi = 2\pi - \tan^{-1}\left(\tfrac{5\sqrt{3}}{5}\right) = 2\pi - \tan^{-1}\left(\sqrt{3}\right) = 2\pi - \tfrac{\pi}{3} = \tfrac{5}{3}\pi\,.$$

Therefore the amplitude is $a = 10$ and the phase is $\phi = \frac{5}{3}\pi$.

Remark. There are many ways to express ϕ. For example, because ϕ is in the fourth quadrant we know that $\frac{3\pi}{2} < \phi < 2\pi$. Using either 2π or $\frac{3\pi}{2}$ as a reference we have

$$\phi = 2\pi - \tan^{-1}\left(\tfrac{5\sqrt{3}}{5}\right) = 2\pi - \tfrac{\pi}{3}\,, \qquad \phi = \tfrac{3\pi}{2} + \tan^{-1}\left(\tfrac{5}{5\sqrt{3}}\right) = \tfrac{3\pi}{2} + \tfrac{\pi}{6}\,,$$
$$\phi = 2\pi - \sin^{-1}\left(\tfrac{5\sqrt{3}}{10}\right) = 2\pi - \tfrac{\pi}{3}\,, \qquad \phi = \tfrac{3\pi}{2} + \sin^{-1}\left(\tfrac{5}{10}\right) = \tfrac{3\pi}{2} + \tfrac{\pi}{6}\,,$$
$$\phi = 2\pi - \cos^{-1}\left(\tfrac{5}{10}\right) = 2\pi - \tfrac{\pi}{3}\,, \qquad \phi = \tfrac{3\pi}{2} + \cos^{-1}\left(\tfrac{5\sqrt{3}}{10}\right) = \tfrac{3\pi}{2} + \tfrac{\pi}{6}\,.$$

The first column uses 2π as the reference while the second uses $\frac{3\pi}{2}$ as the reference. Other inverse trigonometric functions could have been used. Only one correct answer (with no wrong answers) was required for full credit.

Remark. This simple harmonic motion has frequency 2 and period $\frac{2\pi}{2} = \pi$.

(2) [2] The displacement $h(t)$ of a spring-mass system is governed by

$$\ddot{h} + 2\eta\dot{h} + 49h = f(t)\,,$$

where $\eta \geq 0$ is the damping rate and $f(t)$ is a forcing. For what values of η is the system over damped?

Solution. The system is over damped when $\omega_o < \eta$. Because the natural frequency of this system is $\omega_o = \sqrt{49} = 7$, the system is over damped when

$$7 < \eta\,.$$

Alternative Solution. The system is over damped when the associated characteristic polynomial has two real roots. Because the associated characteristic polynomial is

$$p(\zeta) = \zeta^2 + 2\eta\zeta + 49 = (\zeta + \eta)^2 + 49 - \eta^2\,,$$

it has two real roots when $49 - \eta^2 < 0$. Therefore the system is over damped when

$$7 < \eta\,.$$

Remark. You should be able to answer a similar question about when the system is undamped, under damped, or critically damped.

(3) [5] Compute the Green function for the differential operator $\mathrm{L} = \mathrm{D}^2 + 6\mathrm{D} + 13$.

Solution. The Green function $g(t)$ for L solves the initial-value problem

$$g'' + 6g' + 13g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The associated characteristic polynomial is

$$p(\zeta) = \zeta^2 + 6\zeta + 13 = (\zeta + 3)^2 + 2^2\,,$$

which has roots $-3 \pm i2$. Therefore a general solution of the equation is

$$g(t) = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)\,.$$

Because $g(0) = c_1$, the initial condition $g(0) = 0$ implies that $c_1 = 0$. Therefore

$$g(t) = c_2 e^{-3t}\sin(2t)\,,$$
$$g'(t) = 2c_2 e^{-3t}\cos(2t) - 3c_2 e^{-3t}\sin(2t)\,.$$

Because $g'(0) = 2c_2$, the initial condition $g'(0) = 1$ implies that $c_2 = \frac{1}{2}$. Therefore the Green function for L is

$$g(t) = \tfrac{1}{2} e^{-3t}\sin(2t)\,.$$

Remark. For any initial time t_I and any forcing $f(t)$ the *Green Function Formula* gives the solution of the second-order initial-value problem

$$\mathrm{L}v = f(t)\,, \qquad v(t_I) = 0\,, \quad v'(t_I) = 0\,,$$

by

$$v(t) = \int_{t_I}^{t} g(t-s)\, f(s)\, \mathrm{d}s\,.$$

This can serve as a particular solution of $\mathrm{L}y = f(t)$. For $\mathrm{L} = \mathrm{D}^2 + 6\mathrm{D} + 13$ it gives the particular solution

$$y_P(t) = \tfrac{1}{2}\int_{t_I}^{t} e^{-3(t-s)}\sin\big(2(t-s)\big)\, f(s)\, \mathrm{d}s\,.$$

By using the facts that $e^{-3(t-s)} = e^{-3t}e^{3s}$ and that

$$\sin\big(2(t-s)\big) = \sin(2t)\cos(2s) - \cos(2t)\sin(2s)\,,$$

we obtain

$$y_P(t) = \tfrac{1}{2} e^{-3t}\sin(2t)\int_{t_I}^{t} e^{3s}\cos(2s)\, f(s)\, \mathrm{d}s - \tfrac{1}{2} e^{-3t}\cos(2t)\int_{t_I}^{t} e^{3s}\cos(2s)\, f(s)\, \mathrm{d}s\,.$$

The Green Function Formula thereby reduces the problem of finding an explicit particular solution to the evaluation of the two integrals

$$\int_{t_I}^{t} e^{3s}\cos(2s)\, f(s)\, \mathrm{d}s\,, \qquad \int_{t_I}^{t} e^{3s}\cos(2s)\, f(s)\, \mathrm{d}s\,.$$

Even when they can be evaluated, the evaluation can be laborious even for simple f.

This approach should not be taken when the forcing f has characteristic form because either Key Identity Evaluations, the Zero Degree Formula, or Undetemined Coefficients usually provide a much shorter route to an explicit particular solution!