Quiz 5 Solutions, Math 246, Professor David Levermore
Tuesday, 8 October 2019

(1) [2] Given that e^{-2t} is a particular solution of the equation

$$w'' + 4w' + 20w = 16e^{-2t} ,$$

give a real general solution.

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 + 4\mathrm{D} + 20$. Its characteristic polynomial is $p(z) = z^2 + 4z + 20 = (z + 2)^2 + 4^2$, which has conjugate pair of roots $-2 \pm i4$. Therefore a real general solution of the associated homogeneous equation is

$$w_H(t) = c_1 e^{-2t} \cos(4t) + c_2 e^{-2t} \sin(4t) .$$

Because we are given that a particular solution is $w_P(t) = e^{-2t}$, a real general solution of the nonhomogeneous equation is

$$w(t) = c_1 e^{-2t} \cos(4t) + c_2 e^{-2t} \sin(4t) + e^{-2t} .$$

Remark. The forcing $16e^{-2t}$ has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = -2$, and multiplicity $m = 0$. Therefore you should be able to find a particular solution of the equation by using either Key Identity Evaluations, the Zero Degree Formula, Undetermined Coefficients, or the Green Function. Try all four methods! Do they give the same result?

(2) [3] Give the degree, characteristic, and multiplicity for the forcing term of the equation

$$u'' + 4u' + 20u = 5t^3 e^{-2t} \sin(4t) .$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 + 4\mathrm{D} + 20$. Its characteristic polynomial is $p(z) = z^2 + 4z + 20 = (z + 2)^2 + 4^2$, which has conjugate pair of roots $-2 \pm i4$.

The forcing term $5t^3 e^{-2t} \sin(4t)$ has degree $d = 3$, characteristic $\mu + i\nu = -2 + i4$, and multiplicity $m = 1$.

(3) [5] Find a particular solution of the equation

$$v'' - 8v' + 16v = 32e^{4t} .$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 - 8\mathrm{D} + 16$. Its characteristic polynomial is $p(z) = z^2 - 8z + 16 = (z - 4)^2$, which has the double root 4.

Its forcing has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = 4$, and multiplicity $m = 2$. A particular solution $v_P(t)$ should be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Below we show that each of these methods gives the particular solution

$$v_P(t) = 16t^2 e^{4t} .$$

Of these, only the first two had been covered before the quiz.

Key Identity Evaluations. Because $d = 0$ and $m = 2$, we need to evaluate the second derivative of the Key Identity at $z = \mu + i\nu = 4$. Because $p(z) = z^2 - 8z + 16$, the Key Identity and its first two derivatives with respect to z are

$$\begin{aligned}
\mathrm{L}(e^{zt}) &= (z^2 - 8z + 16)e^{zt}\,, \\
\mathrm{L}(t\,e^{zt}) &= (z^2 - 8z + 16)t\,e^{zt} + (2z - 8)e^{zt}\,, \\
\mathrm{L}(t^2 e^{zt}) &= (z^2 - 8z + 16)t^2 e^{zt} + 2(2z - 8)t\,e^{zt} + 2e^{zt}\,.
\end{aligned}$$

By evaluating the second derivative of the Key Identity at $z = 4$ we obtain

$$\mathrm{L}(t^2 e^{4t}) = (4^2 - 8 \cdot 4 + 16)t^2 e^{4t} + 2(2 \cdot 4 - 8)t\,e^{4t} + 2e^{4t} = 2e^{4t}\,.$$

Therefore a particular solution of $\mathrm{L}(v) = 32e^{4t}$ is

$$v_P(t) = 16t^2 e^{4t}\,.$$

Zero Degree Formula. This formula can be used because $d = 0$. For a forcing in the phasor form

$$f(t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)\,,$$

it gives the particular solution

$$v_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right)\,.$$

For this problem $f(t) = 32e^{4t}$ and $p(z) = z^2 - 8z + 20$, so that $\mu + i\nu = 4$, $\alpha - i\beta = 32$, $m = 2$, and $p''(z) = 2$, whereby

$$v_P(t) = t^2 e^{4t}\,\frac{32}{p''(4)} = \frac{32}{2}\,t^2 e^{4t} = 16t^2 e^{4t}\,.$$

Undetermined Coefficients. Because $m + d = 2$, $m = 2$, and $\mu + i\nu = 4$, there is a particular solution of $\mathrm{L}(v) = 32e^{4t}$ in the form

$$v_P(t) = At^2 e^{4t}\,.$$

By taking derivatives we get

$$\begin{aligned}
v_P'(t) &= At^2 \cdot 4e^{4t} + A2t \cdot e^{4t} \\
&= 4At^2 e^{4t} + 2At\,e^{4t}\,, \\
v_P''(t) &= At^2 \cdot 16e^{4t} + 2A2t \cdot 4e^{4t} + A2 \cdot e^{4t} \\
&= 16At^2 e^{4t} + 16At\,e^{4t} + 2A\,e^{4t}\,,
\end{aligned}$$

whereby

$$\begin{aligned}
\mathrm{L}\big(v_P(t)\big) &= v_P''(t) - 8v_P'(t) + 16v_P(t) \\
&= \big[16At^2 e^{4t} + 16At\,e^{4t} + 2A\,e^{4t}\big] - 8\big[4At^2 e^{4t} + 2At\,e^{4t}\big] + 16At^2 e^{4t} \\
&= \big[16 - 8 \cdot 4 + 16\big]At^2 e^{4t} + \big[16 - 8 \cdot 2\big]At\,e^{4t} + 2Ae^{4t} = 2Ae^{4t}\,.
\end{aligned}$$

By setting $2Ae^{4t} = 32e^{4t}$ we see that $2A = 32$, whereby $A = 16$. Therefore a particular solution of $\mathrm{L}(v) = 32e^{4t}$ is

$$v_P(t) = 16t^2 e^{4t}\,.$$

Remark. Because a real general solution of the associated homogeneous equation is

$$v_H(t) = c_1 e^{4t} + c_2 t\, e^{4t}\,,$$

a real general solution of the nonhomogeneous equation $\mathrm{L}(v) = 32e^{4t}$ is

$$v(t) = c_1 e^{4t} + c_2 t\, e^{4t} + 16t^2 e^{4t}\,.$$

Remark. Because the equation has constant coefficients, we can find a particular solution using the Green function. This is not generally advisable because the three methods shown above are generally more efficient. But for this particular equation this is not the case.

Green Function. The Green function $g(t)$ for the operator $\mathrm{L} = \mathrm{D}^2 - 8\mathrm{D} + 16$ satisfies the initial-value problem

$$g'' - 8g' + 16g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

By using the real general solution found above, we know that

$$g(t) = c_1 e^{4t} + c_2 t\, e^{4t}\,.$$

The initial condition $g(0) = 0$ implies $c_1 = 0$, whereby

$$g'(t) = c_2\big(e^{4t} + 4t\, e^{4t}\big)\,.$$

The initial condition $g'(0) = 1$ implies $c_2 = 1$, whereby the Green function is

$$g(t) = t\, e^{4t}\,.$$

The Green function formula with $t_I = 0$ and $f(t) = 32e^{4t}$ gives the particular solution

$$\begin{aligned} v_P(t) &= \int_{t_I}^{t} g(t-s)\, f(s)\, \mathrm{d}s \\ &= \int_0^t (t-s)\, e^{4(t-s)}\, 32e^{4s}\, \mathrm{d}s \\ &= 32e^{4t} \int_0^t (t-s)\, \mathrm{d}s = -32e^{4t}\tfrac{1}{2}(t-s)^2\Big|_{s=0}^{t} = 16t^2 e^{4t}\,. \end{aligned}$$

The Green function formula generally leads to much more complicated integrals.