Quiz 4 Solutions, Math 246, Professor David Levermore
Tuesday, 1 October 2019

(1) [3] Determine the interval of definition for the solution to the initial-value problem

$$u''' + 7t\,u'' - \frac{1}{\sin(t)}\,u' - \frac{\sin(3t)}{5+t}\,u = \frac{e^t}{3-t}\,, \qquad u(-4) = u'(-4) = u''(-4) = 2\,.$$

Solution. This nonhomogeneous linear equation for u is already in normal form. Notice that

- ◇ the coefficient of u'' is continuous everywhere;
- ◇ the coefficient of u' is undefined at $t = n\pi$ for every integer n and is continuous elsewhere;
- ◇ the coefficient of u is undefined at $t = -5$ and is continuous elsewhere;
- ◇ the forcing is undefined at $t = 3$ and is continuous elsewhere;
- ◇ the initial time is $t = -4$.

Therefore the interval of definition is $(-5, -\pi)$ because

- the initial time -4 is in $(-5, -\pi)$,
- all the coefficients and the forcing are continuous over $(-5, -\pi)$,
- the coefficient of u is undefined at $t = -5$,
- the coefficient of u' is undefined at $t = -\pi$.

Remark. All four reasons must be given for full credit.

- ○ The first two are why a (unique) solution exists over the interval $(-5, -\pi)$.
- ○ The last two are why this solution does not exist over a larger interval.

(2) [3] Compute the Wronskian $\mathrm{Wr}[V_1, V_2](t)$ of the functions $V_1(t) = e^{2t}$ and $V_2(t) = t\,e^{2t}$. (Evaluate the determinant and simplify.)

Solution. Because $V_1'(t) = 2e^{2t}$ and $V_2'(t) = e^{2t} + 2t\,e^{2t}$, the Wronskian is

$$\mathrm{Wr}[V_1, V_2](t) = \det\begin{pmatrix} V_1(t) & V_2(t) \\ V_1'(t) & V_2'(t) \end{pmatrix} = \det\begin{pmatrix} e^{2t} & t\,e^{2t} \\ 2e^{2t} & e^{2t} + 2t\,e^{2t} \end{pmatrix}$$
$$= e^{2t}\big(e^{2t} + 2t\,e^{2t}\big) - 2e^{2t}\big(t\,e^{2t}\big) = e^{4t} + 2t\,e^{4t} - 2t\,e^{4t} = e^{4t}\,.$$

Remark. Because $\mathrm{Wr}[V_1, V_2](t) = e^{4t} \neq 0$, we can conclude that the functions $V_1(t) = e^{2t}$ and $V_2(t) = t\,e^{2t}$ are linearly independent.

Remark. It is easily checked that $V_1(t) = e^{2t}$ and $V_2(t) = t\,e^{2t}$ are solutions of $v'' - 4v' + 4v = 0$. Becasue they are linearly independent, they comprise a fundamental set of solutions for the homogeneous second-order equation $v'' - 4v' + 4v = 0$.

Remark. Because $V_1(t) = e^{2t}$ and $V_2(t) = t\,e^{2t}$ solve the homogeneous second-order equation $v'' - 4v' + 4v = 0$, the Abel Theorem states that $w(t) = \mathrm{Wr}[V_1, V_2](t)$ should solve the homogeneous first-order equation $w' - 4w = 0$. This gives a check on the Wronskian calculation above, because e^{4t} solves $w' - 4w = 0$.

(3) [4] Given that e^{2t} and $t\,e^{2t}$ are linearly independent solutions of $v'' - 4v' + 4v = 0$, solve the general initial-value problem associated with $t = 0$ — namely, solve

$$v'' - 4v' + 4v = 0\,, \qquad v(0) = v_0\,, \quad v'(0) = v_1\,.$$

Solution. This is a homogeneous linear equation with constant coefficients. Because we are given that e^{2t} and $t\,e^{2t}$ are solutions to it, we can use the method of linear superposition to seek the solution of the general initial-value problem in the form

$$v(t) = c_1\,e^{2t} + c_2\,t\,e^{2t}\,.$$

Then $v'(t) = c_1 2e^{2t} + c_2(e^{2t} + 2t\,e^{2t})$ and the initial conditions yield

$$v_0 = v(0) = c_1\,, \qquad\qquad v_1 = v'(0) = 2c_1 + c_2\,.$$

It follows that

$$c_1 = v_0\,, \qquad c_2 = v_1 - 2v_0\,.$$

Therefore the solution of the general initial-value problem is

$$v = v_0\,e^{2t} + (v_1 - 2v_0)\,t\,e^{2t}\,.$$

Remark. The solution of the general initial-value problem found above can be written as

$$v = v_0\,(1 - 2t)\,e^{2t} + v_1\,t\,e^{2t}\,.$$

We can read off from this that the natural fundamental set of solutions assosciated with the initial time $t = 0$ is

$$N_0(t) = (1 - 2t)\,e^{2t}\,, \qquad N_1(t) = t\,e^{2t}\,.$$

Notice that these satisfy the initial conditions

$$N_0(0) = 1\,, \qquad\qquad N_1(0) = 0\,,$$
$$N_0'(0) = 0\,, \qquad\qquad N_1'(0) = 1\,.$$