Quiz 3 Solutions, Math 246, Professor David Levermore
Tuesday, 17 September 2019

(1) [4] Consider the solution $x(t)$ of the initial-value problem

$$\ddot{x} = -2x^3\,, \qquad x(0) = 2\,, \quad \dot{x}(0) = -3\,.$$

Find the reduced equation satisfied by $x(t)$.

Solution. This is an initial-value problem with a second-order autonomous equation. Its auxiliary equation is

$$v\,\frac{\mathrm{d}v}{\mathrm{d}x} = -2x^3\,.$$

This first-order equation is separable. Its separated differential form is

$$v\,\mathrm{d}v = -2x^3\,\mathrm{d}x\,.$$

Integrating this gives

$$\tfrac{1}{2}v^2 = -\tfrac{1}{2}x^4 + c\,.$$

The initial conditions imply that $v = -3$ when $x = 2$. Therefore

$$\tfrac{1}{2}(-3)^2 = -\tfrac{1}{2}2^4 + c\,,$$

which yields $c = \frac{1}{2}(9 + 16) = \frac{25}{2}$. Therefore an implicit solution is

$$v^2 = 25 - x^4\,.$$

Because $v = -3$ when $x = 2$, the explicit solution is

$$v = -\sqrt{25 - x^4}\,.$$

Therefore the reduced equation satisfied by $x(t)$ is

$$\dot{x} = -\sqrt{25 - x^4}\,.$$

Remark. In fact, $x(t)$ satisfies the first-order initial-value problem

$$\dot{x} = -\sqrt{25 - x^4}\,, \qquad x(0) = 2\,.$$

This problem cannot be solved in terms of elementary functions.

(2) [2] Suppose we have used the Runge-Kutta method to approximate the solution of an initial-value problem over the time interval $[3, 13]$ with 1000 uniform time steps. About how many uniform time steps do we need to reduce the global error of our approximation by a factor of $\frac{1}{625}$?

Solution. Because the Runge-Kutta method is *fourth order*, its error scales like h^4. To reduce the error by a factor of $\frac{1}{625}$, we must reduce h by a factor of $(\frac{1}{625})^{\frac{1}{4}} = \frac{1}{5}$. Therefore we must increase the number of time steps by a factor of 5, which means that we need $5,000$ uniform time steps.

Remark. You should be able to answer similar questions about the explicit Euler, Runge-trapezoidal, and Runge-midpoint methods.

(3) [4] Consider the initial-value problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 3v - v^2\,, \qquad v(0) = 2\,.$$

Approximate $v(.2)$ using one step of the Runge-trapezoidal method.
Leave your answer as an arithmetic expression.

Solution. Let $f(v) = 3v - v^2$. Set $v_0 = v(0) = 2$.
Then one step of the Runge-trapezoidal method with $h = .2$ yields

$$\begin{aligned}
f_0 &= f(v_0) = 3v_0 - v_0^2 && \text{evaluate } f(v) \text{ at step zero}\\
&\qquad\quad = 3\cdot 2 - 2^2 = 6 - 4 = 2\,, \\
\tilde{v}_1 &= v_0 + hf_0 && \text{full step by explicit Euler}\\
&= 2 + .2\cdot 2 = 2 + .4 = 2.4\,, \\
\tilde{f}_1 &= f(\tilde{v}_1) = 3\tilde{v}_1 - \tilde{v}_1^2 && \text{evaluate } f(v) \text{ at the full step}\\
&\qquad\quad = 3\cdot 2.4 - (2.4)^2\,, \\
v_1 &= v_0 + \tfrac{h}{2}\big[f_0 + \tilde{f}_1\big] && \text{full step by Runge-trapezoidal}\\
&= 2 + .1\big[2 + \big(3\cdot 2.4 - (2.4)^2\big)\big]\,.
\end{aligned}$$

Therefore $v(.2) \approx v_1 = 2 + .1\big[2 + \big(3\cdot 2.4 - (2.4)^2\big)\big]$.

Remark. You do not have to evaluate the above arithemic expression for full credit. It evaluates to $v(.2) \approx v_1 = 2.344$.

Remark. You should be able to answer similar questions using either one step of the Runge-midpoint method or two steps of the explicit Euler method.

Remark. Because the equation is autonomous, its phase-line portrait will show the qualitative behavior of its solutions, and thereby can be used to check your numerical answer.

Remark. This autonomous equation can be solved explicitly.