Quiz 2 Solutions, Math 246, Professor David Levermore
Tuesday, 10 September 2019

(1) [5] Sketch the phase-line portrait for the equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{(w+4)(w+1)^3(w-5)^2}{(1+w^2)^2(w-2)} \,.$$

(a) [3] Identify each stationary point as being either stable, unstable, or semistable. (You do not have to find the solution!)

(b) [2] How does $w(t)$ behave as $t \to \infty$ if $w(3) = 0$? if $w(-3) = 4$?

Solution (a). This equation is autonomous. Its right-hand side is undefined at $w = 2$ and is differentiable elsewhere. Its stationary points are found by setting

$$\frac{(w+4)(w+1)^3(w-5)^2}{(1+w^2)^2(w-2)} = 0 \,.$$

Therefore the stationary points are $w = -4$, $w = -1$, and $w = 5$. A sign analysis of the right-hand side shows that the phase-line portrait is

$$\begin{array}{ccccccccccc}
 & - & & + & & - & & + & & + & \\
 & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \circ & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & w \\
 & & -4 & & -1 & & 2 & & 5 & & \\
 & & \text{unstable} & & \text{stable} & & \text{undefined} & & \text{semistable} & &
\end{array}$$

Remark. Here the terms stable, unstable, and semistable are applied to solutions. The point $w = 2$ is not a solution, so these terms should not be applied to it.

Solution (b). Because the initial condition $w(3) = 0$ has the initial value 0, which lies in the interval $(-1, 2)$, the phase-line portrait shows that

$$\lim_{t\to\infty} w(t) = -1 \,.$$

Because the initial condition $w(-3) = 4$ has the initial value 4, which lies in the interval $(2, 5)$, the phase-line portrait shows that

$$\lim_{t\to\infty} w(t) = 5 \,.$$

(2) [5] A tank with a capacity of 25 liters initially contains 13 liters of brine (salt water) with a salt concentration of 3 grams per liter (gr/lit). At time $t = 0$ brine with a salt concentration of 5 grams per liter (gr/lit) begins to flow into the tank at a constant rate of 4 liters per minute (lit/min) and the well-stirred mixture flows out of the tank at a constant rate of 2 liters per minute (lit/min). Write down an initial-value problem that governs the grams of salt in the tank for $t > 0$ until the tank overflows. (Do not solve the initial-value problem!)

Solution. Let $S(t)$ be the amount (gr) of salt and $V(t)$ be the volume (lit) of brine in the tank at time t minutes. We have the following (optional) picture.

inflow 4 lit/min $\longrightarrow$ 5 gr/lit	brine mass $S(t)$ gr brine volume $V(t)$ lit concentration $C(t)$ gr/lit capacity 25 lit	$\longrightarrow$ outflow 2 lit/min	initial conditions $C(0) = 3$ gr/lit, $V(0) = 13$ lit.

We want to write down an initial-value problem that governs $S(t)$.

Because $V(0) = 13$ and brine flow in at a rate of 4 lit/min while it flows out at a rate of 2 lit/min, we see that $V(t) = 13 + 2t$ lit. Because the mixture is well-stirred, we have $C(t) = S(t)/V(t)$ gr/lit. Therefore the rate at which $S(t)$ changes will be

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= (\text{concentration in}) \cdot (\text{flow rate in}) - (\text{concentration out}) \cdot (\text{flow rate out}) \\ &= 5 \cdot 4 - C \cdot 2 = 20 - \frac{S}{V} \cdot 2 = 20 - \frac{2}{13 + 2t}\, S \quad \text{gr/min}\,.\end{aligned}$$

Because $S(0) = C(0) \cdot V(0) = 3 \cdot 13 = 39$ gr, the initial-value problem governing S is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 20 - \frac{2}{13 + 2t}\, S\,, \qquad S(0) = 39\,.$$