

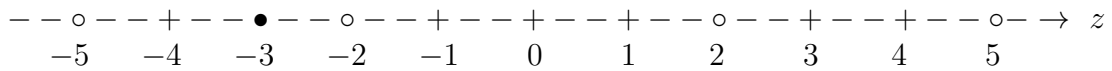
Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 3 September 2019

(1) [2] What is the interval of definition for the solution of the initial-value problem

$$\frac{dw}{dz} + \frac{e^z}{z^2 - 25} w = \frac{\sin(z)}{z^2 - 4}, \quad w(-3) = 7.$$

(You do not need to solve the differential equation, but you must give your reasoning!)

Solution. This is a nonhomogeneous linear equation that is already in normal form. The coefficient $e^z/(z^2 - 25)$ is undefined at $z = \pm 5$ and is continuous elsewhere. The forcing $\sin(z)/(z^2 - 4)$ is undefined at $z = \pm 2$ and is continuous elsewhere. The initial time is $z = -3$. This can be pictured on the z -axis as follows.



Therefore the interval of definition for the solution is $(-5, -2)$ because:

- the initial time $z = -3$ is in $(-5, -2)$,
- the coefficient and forcing are both continuous over $(-5, -2)$,
- the coefficient is undefined at $z = -5$,
- the forcing is undefined at $z = -2$.

(2) [4] Solve the initial-value problem

$$(1 + t^2) \frac{dv}{dt} + 2tv = 8t^3, \quad v(0) = 3.$$

Solution. This is a nonhomogeneous linear equation. Its normal form is

$$\frac{dv}{dt} + \frac{2t}{1 + t^2} v = \frac{8t^3}{1 + t^2}.$$

The coefficient and forcing are continuous everywhere. Therefore the interval of definition will be $(-\infty, \infty)$.

An integrating factor is $e^{A(t)}$ where $A'(t) = 2t/(1 + t^2)$. Setting $A(t) = \log(1 + t^2)$, we obtain $e^{A(t)} = (1 + t^2)$. Hence, the equation has the integrating factor form

$$\frac{d}{dt}((1 + t^2)v) = (1 + t^2) \cdot \frac{8t^3}{1 + t^2} = 8t^3.$$

Integrating both sides yields

$$(1 + t^2)v = 2t^4 + c.$$

Imposing the initial condition $v(0) = 3$ gives

$$(1 + 0^2)3 = 2 \cdot 0^4 + c,$$

whereby $c = 3$. Therefore the solution of the initial-value problem is

$$v = \frac{2t^4 + 3}{1 + t^2}.$$

Remark. The interval of definition of this solution is indeed $(-\infty, \infty)$.

(3) [4] Find an implicit solution of the initial-value problem

$$\frac{dy}{dx} = e^x \frac{y^2 - 9}{2y}, \quad y(0) = -5.$$

Solution. This is a nonautonomous, separable equation. It is undefined at $y = 0$ and is continuous elsewhere. Because $(y^2 - 9) = (y - 3)(y + 3)$, it has stationary points at $y = \pm 3$. Moreover, it is differentiable at these stationary points. Its separated differential form is

$$\frac{2y}{y^2 - 9} dy = e^x dx,$$

whereby

$$\int \frac{2y}{y^2 - 9} dy = \int e^x dx.$$

Upon integrating both sides we find the implicit general solution

$$\log(|y^2 - 9|) = e^x + c.$$

Imposing the initial condition $y(0) = -5$ implies that

$$\log(|(-5)^2 - 9|) = e^0 + c,$$

whereby $c = \log(16) - 1$. Therefore an implicit solution of the initial-value problem is

$$\log(y^2 - 9) = e^x - 1 + \log(16),$$

which can also be expressed as

$$y^2 - 9 = 16 \exp(e^x - 1).$$

Remark. The differential equation is in normal form and is differentiable at its stationary points, $y = \pm 3$. Hence, because the initial condition is $y(0) = -5$, the solution of the initial-value problem must take values $y(x)$ that lie in the interval $(-\infty, -3)$.

Remark. Because the initial condition is $y(0) = -5$, the explicit solution of the initial-value problem is

$$y = -\sqrt{9 + 16 \exp(e^x - 1)}.$$

Notice that this solution takes values in the interval $(-\infty, -3)$. Notice too that its interval of definition is $(-\infty, \infty)$.