Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 3 September 2019

(1) [2] What is the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{e^z}{z^2 - 25}\, w = \frac{\sin(z)}{z^2 - 4}\,, \qquad w(-3) = 7\,.$$

(You do not need to solve the differential equation, but you must give your reasoning!)

Solution. This is a nonhomogeneous linear equation that is already in normal form. The coefficient $e^z/(z^2 - 25)$ is undefined at $z = \pm 5$ and is continuous elsewhere. The forcing $\sin(z)/(z^2-4)$ is undefined at $z = \pm 2$ and is continuous elsewhere. The initial time is $z = -3$. This can be pictured on the z-axis as follows.

$$\begin{array}{ccccccccccccc} --\circ-- & +-- & \bullet-- & \circ-- & +-- & +-- & +-- & \circ-- & +-- & +-- & \circ- & \rightarrow & z \\ -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & & \end{array}$$

Therefore the interval of definition for the solution is $(-5, -2)$ because:

- the initial time $z = -3$ is in $(-5, -2)$,
- the coefficient and forcing are both continuous over $(-5, -2)$,
- the coefficient is undefined at $z = -5$,
- the forcing is undefined at $z = -2$.

(2) [4] Solve the initial-value problem

$$(1 + t^2)\,\frac{\mathrm{d}v}{\mathrm{d}t} + 2t\, v = 8t^3\,, \qquad v(0) = 3\,.$$

Solution. This is a nonhomogeneous linear equation. Its normal form is

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{2t}{1 + t^2}\, v = \frac{8t^3}{1 + t^2}\,.$$

The coefficient and forcing are continuous everywhere. Therefore the interval of definition will be $(-\infty, \infty)$.

An integrating factor is $e^{A(t)}$ where $A'(t) = 2t/(1 + t^2)$. Setting $A(t) = \log(1 + t^2)$, we obtain $e^{A(t)} = (1 + t^2)$. Hence, the equation has the intgrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1 + t^2)v\big) = (1 + t^2) \cdot \frac{8t^3}{1 + t^2} = 8t^3\,.$$

Integrating both sides yields

$$(1 + t^2)v = 2t^4 + c\,.$$

Imposing the initial condition $v(0) = 3$ gives

$$(1 + 0^2)3 = 20^4 + c\,,$$

whereby $c = 3$. Therfore the solution of the initial-value problem is

$$v = \frac{2t^4 + 3}{1 + t^2}\,.$$

Remark. The interval of definition of this solution is indeed $(-\infty, \infty)$.

(3) [4] Find an implicit solution of the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^x \, \frac{y^2 - 9}{2y}\,, \qquad y(0) = -5\,.$$

Solution. This is a nonautonomous, separable equation. It is undefind at $y = 0$ and is continuous elsewhere. Because $(y^2 - 9) = (y - 3)(y + 3)$, it has stationary points at $y = \pm 3$. Moreover, it is differentiable at these stationary points. Its separated differential form is

$$\frac{2y}{y^2 - 9}\,\mathrm{d}y = e^x\,\mathrm{d}x\,,$$

whereby

$$\int \frac{2y}{y^2 - 9}\,\mathrm{d}y = \int e^x\,\mathrm{d}x\,.$$

Upon integrating both sides we find the implicit general solution

$$\log(|y^2 - 9|) = e^x + c\,.$$

Imposing the initial condition $y(0) = -5$ implies that

$$\log(|(-5)^2 - 9|) = e^0 + c\,,$$

whereby $c = \log(16) - 1$. Therefore an implicit solution of the initial-vaqlue problem is

$$\log(y^2 - 9) = e^x - 1 + \log(16)\,,$$

which can also be expressed as

$$y^2 - 9 = 16\exp(e^x - 1)\,.$$

Remark. The differential equation is in normal form and is differentiable at its stationary points, $y = \pm 3$. Hence, because the initial condition is $y(0) = -5$, the solution of the initial-value problem must take values $y(x)$ that lie in the interval $(-\infty, -3)$.

Remark. Because the initial condition is $y(0) = -5$, the explicit solution of the initial-value problem is

$$y = -\sqrt{9 + 16\exp(e^x - 1)}\,.$$

Notice that this solution takes values in the interval $(-\infty, -3)$. Notice too that its interval of definition is $(-\infty, \infty)$.