Math 246, Professor David Levermore
Group Work Exercises for Discussion 10
Monday, 4 November 2019

Answers to the following exercises should be worked out on the board space for your group.
Your reasoning must be shown for full credit!

First Set of Group Work Exercises [4]

Use the table on page 2. Consider the initial-value problem

$$y'' + 16y = f(t)\,, \qquad y(0) = -2\,, \quad y'(0) = 5\,.$$

where

$$f(t) = \begin{cases} t & \text{for } 0 \le t < 3\,, \\ 6 - t & \text{for } 3 \le t < 6\,, \\ 0 & \text{for } 6 \le t\,. \end{cases}$$

(1) Compute $F(s) = \mathcal{L}[f](s)$.
(2) Compute $Y(s) = \mathcal{L}[y](s)$.
(3) Compute $y(t)$.
(4) Solve the initial-value problem with $f(t)$ replaced by $9\,\delta(t-3)$.

Second Set of Group Work Exercises [3]

Two interconnected tanks contain brine (salt water). At $t = 0$ the first tank contains 23 liters and the second contains 32 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 17 grams of salt in the first tank and 29 grams in the second.

(1) Give an initial-value problem that governs how many grams of salt are in each tank as a function of time.
(2) Give the interval of definition for the solution of the above initial-value problem.
(3) Express the above initial-value problem in the form

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,, \qquad \mathbf{x}(0) = \mathbf{x}^I\,.$$

Give $\mathbf{A}(t)$, $\mathbf{f}(t)$, and $\mathbf{x}^I$.

Third Set of Group Work Exercises [3]

Consider the matrix-valued function

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} 1 & -2t^2 \\ t^2 & 4 - t^4 \end{pmatrix}\,.$$

(1) Compute $\det\big(\boldsymbol{\Psi}(t)\big)$.
(2) Compute $\boldsymbol{\Psi}(t)^{-1}$.
(3) Compute $\boldsymbol{\Psi}(t)^{-1}\boldsymbol{\Psi}'(t)$.

Warning. Quiz 8 will ask you to recast a higher-order equation as a first-order system of ordinary differential equations.

Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[j'(t)](s) = sJ(s) - j(0) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s),\ c \ge 0,$$
$$\text{and } u \text{ is the unit step function}\,.$$

$$\mathcal{L}[\delta(t-c)h(t)](s) = e^{-cs}h(c) \qquad \text{where } c \ge 0 \text{ and } \delta \text{ is the unit impulse}\,.$$

Remark. This is the table that will be provided on Exam 3.