Solutions to Final Exam Sample Problems, Math 246, Fall 2019

(1) Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t} = (9 - y^2)y^2$.

(a) Find all of its stationary points and classify their stability.
(b) Sketch its phase-line portrait in the interval $-5 \le y \le 5$.
(c) If $y_1(0) = -1$, how does the solution $y_1(t)$ behave as $t \to \infty$?
(d) If $y_2(0) = 4$, how does the solution $y_2(t)$ behave as $t \to \infty$?
(e) Evaluate

$$\lim_{t\to\infty} \big(y_2(t) - y_1(t)\big) .$$

Solution (a,b). The right-hand side factors as $(3 + y)(3 - y)y^2$. The stationary solutions are $y = -3$, $y = 0$, and $y = 3$. Therefore a sign analysis of $(3 + y)(3 - y)y^2$ shows that the phase-line portrait for this equation is

$$\begin{array}{ccccccccc} & - & & + & & + & & - & \\ \leftarrow\leftarrow\leftarrow\leftarrow & & \bullet \rightarrow\rightarrow\rightarrow\rightarrow & & \bullet \rightarrow\rightarrow\rightarrow\rightarrow & & \bullet \leftarrow\leftarrow\leftarrow\leftarrow & & y \\ & & -3 & & 0 & & 3 & & \\ & & \text{unstable} & & \text{semistable} & & \text{stable} & & \end{array}$$

Solution (c). The phase-line shows that if $y_1(0) = -1$ then $y_1(t) \to 0$ as $t \to \infty$.

Solution (d). The phase-line shows that if $y_2(0) = 4$ then $y_2(t) \to 3$ as $t \to \infty$.

Solution (e). The answers to parts (c) and (d) show that

$$\lim_{t\to\infty} \big(y_2(t) - y_1(t)\big) = \lim_{t\to\infty} y_2(t) - \lim_{t\to\infty} y_1(t) = 3 - 0 = 3 .$$

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(2) Solve each of the following initial-value problems and give the interval of definition of each solution.

(a) $x' = \dfrac{t}{(t^2 + 1)x}$, $\qquad x(0) = -3$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}t} + \dfrac{2ty}{1 + t^2} = t^2$, $\quad y(0) = 1$.

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{e^x y + 2x}{2y + e^x} = 0$, $\quad y(0) = 0$.

Solution (a). This is a *nonautonomous, separable* equation. It is undefined when $x = 0$. Its separated form is

$$x\,\mathrm{d}x = \frac{t}{t^2 + 1}\,\mathrm{d}t .$$

Integrate this to obtain

$$\tfrac{1}{2}x^2 = \tfrac{1}{2}\log(t^2 + 1) + c .$$

The initial condition $x(0) = -3$ gives $\frac{1}{2}(-3)^2 = \frac{1}{2}\log(1) + c$, which implies that $c = \frac{9}{2}$. Therefore the solution is governed implicitly by

$$x^2 = \log(t^2 + 1) + 9 .$$

Hence, because $x(0) = -3 < 0$, the explicit solution of the initial-value problem is

$$x = -\sqrt{\log(t^2+1) + 9}\,.$$

Because $\log(t^2+1) > 0$ for every t, the interval of definition for this solution is $(-\infty, \infty)$.

Solution (b). This is a *nonhomogeneous, linear* equation that is already in normal form. An integrating factor is

$$\exp\left(\int_0^t \frac{2s}{1+s^2}\,\mathrm{d}s\right) = \exp\big(\log(1+t^2)\big) = 1+t^2\,,$$

so that the integrating factor form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1+t^2)y\big) = (1+t^2)t^2 = t^2 + t^4\,.$$

Integrate this to obtain

$$(1+t^2)y = \tfrac{1}{3}t^3 + \tfrac{1}{5}t^5 + c\,.$$

The initial condition $y(0) = 1$ implies that $c = (1+0^2)\cdot 1 - \frac{1}{3}0^3 - \frac{1}{5}0^5 = 1$. Therefore

$$y = \frac{1 + \frac{1}{3}t^3 + \frac{1}{5}t^5}{1+t^2}\,.$$

This solution exists for every t, so its interval of definition is $(-\infty, \infty)$.

Remark. Because this equation is linear, we can see that the interval of definition of its solution is $(-\infty, \infty)$ without solving it because both its coefficient and forcing are continuous over $(-\infty, \infty)$.

Solution (c). The initial-value problem is

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{e^x y + 2x}{2y + e^x} = 0\,, \qquad y(0) = 0\,.$$

Express this equation in the differential form

$$(e^x y + 2x)\,\mathrm{d}x + (2y + e^x)\,\mathrm{d}y = 0\,.$$

This differential form is *exact* because

$$\partial_y(e^x y + 2x) = e^x \qquad = \qquad \partial_x(2y + e^x) = e^x\,.$$

Therefore we can find $H(x, y)$ such that

$$\partial_x H(x,y) = e^x y + 2x\,, \qquad \partial_y H(x,y) = 2y + e^x\,.$$

The first equation implies $H(x,y) = e^x y + x^2 + h(y)$. Plugging this into the second equation gives

$$e^x + h'(y) = 2y + e^x\,,$$

which yields $h'(y) = 2y$. Taking $h(y) = y^2$, yields $H(x,y) = e^x y + x^2 + y^2$. Therefore a general solution is

$$e^x y + x^2 + y^2 = c\,.$$

The initial condition $y(0) = 0$ implies that $c = e^0 \cdot 0 + 0^2 + 0^2 = 0$. Therefore

$$y^2 + e^x y + x^2 = 0\,.$$

The quadratic formula then yields the explicit solution

$$y = \frac{-e^x + \sqrt{e^{2x} - 4x^2}}{2} .$$

Here the positive square root is taken because that solution satisfies the initial condition. Its interval of definition is the largest interval (x_L, x_R) containing the initial time 0 over which $e^{2x} > 4x^2$. We cannot find the endpoints of this interval explicitly.

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(3) Determine constants a and b such that the following differential equation is exact. Then find a general solution in implicit form.

$$\big(ye^x + y^3\big)\,\mathrm{d}x + \big(ae^x + bxy^2\big)\,\mathrm{d}y = 0 .$$

Solution. This equation will be exact whenever

$$\partial_y\big(ye^x + y^3\big) = \partial_x\big(ae^x + bxy^2\big) .$$

Because

$$\partial_y\big(ye^x + y^3\big) = e^x + 3y^2 ,$$
$$\partial_x\big(ae^x + bxy^2\big) = ae^x + by^2 ,$$

the equation will be exact whenever $a = 1$ and $b = 3$.

When $a = 1$ and $b = 3$ there exists $H(x, y)$ such that

$$\partial_x H(x, y) = ye^x + y^3 , \qquad \partial_y H(x, y) = e^x + 3xy^2 .$$

The second equation implies $H(x, y) = e^x y + xy^3 + h(x)$. Plugging this into the first equation gives

$$e^x y + y^3 + h'(x) = ye^x + y^3 ,$$

which yields $h'(x) = 0$. Taking $h(x) = 0$ yields $H(x, y) = e^x y + xy^3$. Therefore a general solution is

$$e^x y + xy^3 = c .$$

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(4) Consider the following Matlab function m-file.

```
function [t,y] = solveit(ti, yi, tf, n)
t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = ti; y(1) = yi; h = (tf - ti)/n;
for i = 1:n
t(i + 1) = t(i) + h; y(i + 1) = y(i) + h*((t(i))^4 + (y(i))^2);
end
```

Suppose that the input values are ti = 1, yi = 1, tf = 5, and n = 40.
(a) What initial-value problem is being approximated numerically?
(b) What numerical method is being used?
(c) What is the step size?
(d) What are the output values of t(2), y(2), t(3), and y(3)?

Solution (a). The initial-value problem being approximated numerically is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^4 + y^2 \,, \qquad y(1) = 1\,.$$

Solution (b). The explicit Euler (forward Euler) method is being used.

Solution (c). The step size is

$$\mathrm{h} = \frac{\mathrm{tF} - \mathrm{tI}}{\mathrm{n}} = \frac{5-1}{40} = \frac{4}{40} = \frac{1}{10} = .1\,.$$

Solution (d). By carrying out the "for" loop in the Matlab code for $i = 1$ and $i = 2$ we obtain the output values

$$\begin{aligned}
\mathrm{t}(2) &= \mathrm{t}(1) + \mathrm{h} = 1 + .1 = 1.1\,,\\
\mathrm{y}(2) &= \mathrm{y}(1) + \mathrm{h}{*}((\mathrm{t}(1))\hat{}4 + (\mathrm{y}(1))\hat{}2) = 1 + .1(1^4 + 1^2) = 1 + .1\cdot 2 = 1.2\,.\\
\mathrm{t}(3) &= \mathrm{t}(2) + \mathrm{h} = 1.1 + .1 = 1.2\,,\\
\mathrm{y}(3) &= \mathrm{y}(2) + \mathrm{h}{*}((\mathrm{t}(2))\hat{}4 + (\mathrm{y}(2))\hat{}2) = 1.2 + .1\big((1.1)^4 + (1.2)^2\big)\,.
\end{aligned}$$

You DO NOT have to work out the arithmetic to compute y(3)! If you did then you would obtain $y(3) = 1.49041$.

Remark. You should be able to answer similar questions that employ either the Runge-trapeziodal or Runge-midpoint method.

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(5) Let $y(t)$ be the solution of the initial-value problem

$$y' = 4t(y + y^2)\,, \qquad y(0) = 1\,.$$

(a) Use two steps of the explicit Euler method to approximate $y(1)$.
(b) Use one step of the Runge-trapeziodal method to approximate $y(1)$.
(c) Use one step of the Runge-midpoint method to approximate $y(1)$.

Solution (a). The explicit (forward) Euler method is

$$f_n = f(y_n, t_n)\,, \qquad y_{n+1} = y_n + hf_n\,, \qquad t_{n+1} = t_n + h\,,$$

where h is the time step, t_0 is the initial time, and y_0 is the initial value.

When the explicit Euler method is applied with $h = 0.5$, $t_0 = 0$, $y_0 = 1$, and $f(y,t) = 4t(y + y^2)$ for two steps

$$\begin{aligned}
f_0 &= f(y_0, t_0) = f(1, 0) = 4\cdot 0\cdot(1 + 1^2) = 0\,,\\
y_1 &= y_0 + hf_0 = 1 + 0.5\cdot 0 = 1\,,\\
t_1 &= t_0 + h = 0 + 0.5 = 0.5\,,\\
f_1 &= f(y_1, t_1) = f(1, 0.5) = 4\cdot 0.5\cdot(1 + 1^2) = 4\,,\\
y_2 &= y_1 + hf_1 = 1 + 0.5\cdot 4 = 3\,.
\end{aligned}$$

Therefore the approximation is

$$y(1) \approx y_2 = 3\,.$$

Solution (b). The Runge-trapeziodal method is

$$f_n = f(y_n, t_n)\,, \qquad \tilde{y}_{n+1} = y_n + hf_n\,, \qquad t_{n+1} \ = t_n + h\,,$$
$$\tilde{f}_{n+1} = f(\tilde{y}_{n+1}, t_{n+1})\,, \qquad y_{n+1} = y_n + \tfrac{h}{2}(f_n + \tilde{f}_{n+1})\,.$$

where h is the time step, t_0 is the initial time, and y_0 is the initial value.

When the Runge-trapezoidal method is applied with $h = 1$, $t_0 = 0$, $y_0 = 1$, and $f(y,t) = 4t(y + y^2)$ for one step

$$\begin{aligned} f_0 &= f(y_0, t_0) = f(1,0) = 4 \cdot 0 \cdot (1 + 1^2) = 0\,, \\ \tilde{y}_1 &= y_0 + hf_0 = 1 + 1 \cdot 0 = 1\,, \\ t_1 &= t_0 + h = 0 + 1 = 1\,, \\ \tilde{f}_1 &= f(y_1, t_1) = f(1,1) = 4 \cdot 1 \cdot (1 + 1^2) = 8\,, \\ y_1 &= y_0 + \tfrac{h}{2}(f_0 + \tilde{f}_1) = 1 + 0.5 \cdot (0 + 8) = 5\,. \end{aligned}$$

Therefore the approximation is

$$y(1) \approx y_1 = 5\,.$$

Solution (c). The Runge-midpoint method is

$$f_n = f(y_n, t_n)\,, \qquad y_{n+\frac{1}{2}} = y_n + \tfrac{h}{2} f_n\,, \qquad t_{n+\frac{1}{2}} = t_n + \tfrac{h}{2}\,,$$
$$f_{n+\frac{1}{2}} = f(y_{n+\frac{1}{2}}, t_{n+\frac{1}{2}})\,, \qquad y_{n+1} = y_n + hf_{n+\frac{1}{2}}\,, \qquad t_{n+1} = t_n + h\,.$$

where h is the time step, t_0 is the initial time, and y_0 is the initial value.

When the Runge-midpoint method is applied with $h = 1$, $t_0 = 0$, $y_0 = 1$, and $f(y,t) = 4t(y + y^2)$ for one step

$$\begin{aligned} f_0 &= f(y_0, t_0) = f(1,0) = 4 \cdot 0 \cdot (1 + 1^2) = 0\,, \\ y_{\frac{1}{2}} &= y_0 + \tfrac{h}{2} f_0 = 1 + .5 \cdot 0 = 1\,, \\ t_{\frac{1}{2}} &= t_0 + \tfrac{h}{2} = 0 + 0.5 = 0.5\,, \\ f_{\frac{1}{2}} &= f(y_{\frac{1}{2}}, t_{\frac{1}{2}}) = f(1, 0.5) = 4 \cdot 0.5 \cdot (1 + 1^2) = 4\,, \\ y_1 &= y_0 + hf_{\frac{1}{2}} = 1 + 1 \cdot 4 = 5\,. \end{aligned}$$

Therefore the approximation is

$$y(1) \approx y_1 = 5\,.$$

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(6) Consider the following Matlab commands.

```
[t,y] = ode45(@(t,y) y.*(y−1).*(2−y), [0,3], -0.5:0.5:2.5);
plot(t,y)
```

The following questions need not be answered in Matlab format!

(a) What is the differential equation being solved numerically?
(b) Give the initial condition for each solution being approximated?
(c) Over what time interval are the solutions being approximated?
(d) Sketch each of these solutions over this time interval on a single graph. Label the initial value of each solution clearly.

(e) What is the limiting behavior of each solution as $t \to \infty$?

Solution (a). The differential equation being solved numerically is

$$y' = y(y-1)(2-y)\,.$$

Solution (b). There are seven solutions being approximated numerically. Their initial conditions are

$$y(0) = -0.5\,, \qquad y(0) = 0.0\,, \qquad y(0) = 0.5\,, \qquad y(0) = 1.0\,,$$
$$y(0) = 1.5\,, \qquad y(0) = 2.0\,, \qquad y(0) = 2.5\,.$$

Solution (c). These solutions are being approximated over the time interval $[0, 3]$.

Solutions (d) and (e). A sign analysis of $y(y-1)(2-y)$ shows that the phase-line portrait for this equation is

$$\begin{array}{ccccccc} + & & - & & + & & - \\ \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow\ y \\ & 0 & & 1 & & 2 & \\ & \text{stable} & & \text{unstable} & & \text{stable} & \end{array}$$

This shows that:

for the initial condition $y(0) = -0.5$ $\quad y(t)$ increases to 0 as $t \to \infty$;
for the initial condition $y(0) = 0.0$ $\quad y(t) = 0$ for every $t > 0$;
for the initial condition $y(0) = 0.5$ $\quad y(t)$ decreases to 0 as $t \to \infty$;
for the initial condition $y(0) = 1.0$ $\quad y(t) = 1$ for every $t > 0$;
for the initial condition $y(0) = 1.5$ $\quad y(t)$ increases to 2 as $t \to \infty$;
for the initial condition $y(0) = 2.0$ $\quad y(t) = 2$ for every $t > 0$;
for the initial condition $y(0) = 2.5$ $\quad y(t)$ decreases to 2 as $t \to \infty$.

If someone asks for it, the sketch for part (d) will be given during the review session. It should be fairly evident from the above analysis.

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(7) Suppose we are using the Runge-midpoint method to numerically approximate the solution of an initial-value problem over the time interval $[1, 9]$. By what factor would we expect the error to decrease when we increase the number of time steps taken from 400 to 2000?

Solution. The Runge-midpoint method is second order, which means its (global) error scales like h^2 where h is the step size. When the number of time steps taken increases from 400 to 2000, the step size h decreases by a factor of $\frac{1}{5}$. Therefore the error will decrease (like h^2) by a factor of $(\frac{1}{5})^2 = \frac{1}{25}$.

Remark. You should be able to answer similar questions about the explicit Euler, Runge-trapezoidal, and Runge-Kutta methods.

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(8) A NASA engineer has used the Runge-Kutta method to approximate the solution of an initial-value problem over the time interval $[2, 10]$ with 800 uniform time steps.

(a) How many uniform time steps are needed to reduce the global error by a factor of $\frac{1}{256}$?

(b) What is the order of a numerical method that reduces the global error by a factor of $\frac{1}{256}$ when the step size is halved?

Solution (a). The Runge-Kutta method is fourth order. Because

$$\tfrac{1}{256} = \left(\tfrac{1}{4}\right)^4 ,$$

we see that $4 \cdot 800 = 3200$ time steps are needed.

Solution (b). When the time step is halved the global error of an n^{th} order method is reduced by a factor of

$$\left(\tfrac{1}{2}\right)^n .$$

Because

$$\tfrac{1}{256} = \left(\tfrac{1}{2}\right)^8 ,$$

we see that the method must be eighth order.

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(9) Give an explicit real-valued general solution of the following equations.

(a) $y'' - 2y' + 5y = t\,e^t + \cos(2t)$

(b) $\ddot{u} - 3\dot{u} - 10u = t\,e^{-2t}$

(c) $v'' + 9v = \cos(3t)$

(d) $w'''' + 13w'' + 36w = 9\sin(t)$

Solution (a). This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2 .$$

This has the conjugate pair of roots $1 \pm i2$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) .$$

The forcing $t\,e^t + \cos(2t)$ has composite characteristic form because it is the sum of two terms, each of which has characteristic form. The forcing term te^t has degree $d = 1$, characteristic $\mu + i\nu = 1$, and multiplicity $m = 0$. The forcing term $\cos(2t)$ has degree $d = 0$, characteristic $\mu + i\nu = i2$ and multiplicity $m = 0$. Because their characteristics are different, these terms must be treated separately. A particular solution $y_P(t)$ can be found by either the method of Key Identity Evaluations or the method of Undetermined Coefficients.

Key Indentity Evaluations. The forcing term $t\,e^t$ has degree $d = 1$, characteristic $\mu + i\nu = 1$, and multiplicity $m = 0$. Because $m = 0$ and $m + d = 1$, we need the Key Identity and its first derivative

$$\mathrm{L}(e^{zt}) = (z^2 - 2z + 5)e^{zt} ,$$
$$\mathrm{L}(t\,e^{zt}) = (z^2 - 2z + 5)t\,e^{zt} + (2z - 2)\,e^{zt} .$$

Evaluate these at $z = 1$ to find $\mathrm{L}(e^t) = 4e^t$ and $\mathrm{L}(t\,e^t) = 4t\,e^t$. Dividing the second of these equations by 4 yields $\mathrm{L}(\frac{1}{4}t\,e^t) = t\,e^t$, which implies $y_{P1}(t) = \frac{1}{4}t\,e^t$.

The forcing term $\cos(2t)$ has degree $d = 0$, characteristic $\mu + i\nu = i2$, and multiplicity $m = 0$. Because $m = m + d = 0$, we only need the Key Identity,

$$\mathrm{L}(e^{zt}) = (z^2 - 2z + 5)e^{zt}\,.$$

Evaluating this at $z = i2$ to find $\mathrm{L}(e^{i2t}) = (1 - i4)e^{i2t}$ and dividing by $1 - i4$ yeilds

$$\mathrm{L}\bigg(\frac{e^{i2t}}{1 - i4}\bigg) = e^{i2t}\,.$$

Because $\cos(2t) = \mathrm{Re}(e^{i2t})$, the above equation implies

$$\begin{aligned} y_{P2}(t) &= \mathrm{Re}\bigg(\frac{e^{i2t}}{1 - i4}\bigg) = \mathrm{Re}\bigg(\frac{(1 + i4)e^{i2t}}{1^2 + 4^2}\bigg) \\ &= \tfrac{1}{17}\,\mathrm{Re}\big((1 + i4)e^{i2t}\big) = \tfrac{1}{17}\big(\cos(2t) - 4\sin(2t)\big)\,. \end{aligned}$$

Upon combining these two particular solutions with the general solution of the associated homogeneous problem found earlier we obtain the general solution

$$\begin{aligned} y &= y_H(t) + y_{P1}(t) + y_{P2}(t) \\ &= c_1 e^t\cos(2t) + c_2 e^t\sin(2t) + \tfrac{1}{4}t\,e^t + \tfrac{1}{17}\cos(2t) - \tfrac{4}{17}\sin(2t)\,. \end{aligned}$$

Undetermined Coefficients. The forcing term $t\,e^t$ has degree $d = 1$, characteristic $\mu + i\nu = 1$, and multiplicity $m = 0$. Because $m = 0$ and $m + d = 1$, we seek a particular solution of the form

$$y_{P1}(t) = A_0 t\,e^t + A_1 e^t\,.$$

Because

$$y'_{P1}(t) = A_0 t\,e^t + (A_0 + A_1)e^t\,, \qquad y''_{P1}(t) = A_0 t\,e^t + (2A_0 + A_1)e^t\,,$$

we see that

$$\begin{aligned} \mathrm{L}y_{P1}(t) &= y''_{P1}(t) - 2y'_{P1}(t) + 5y_{P1}(t) \\ &= \big(A_0 t\,e^t + (2A_0 + A_1)e^t\big) - 2\big(A_0 t\,e^t + (A_0 + A_1)e^t\big) \\ &\quad + 5\big(A_0 t\,e^t + A_1 e^t\big) \\ &= 4A_0 t\,e^t + 4A_1 e^t\,. \end{aligned}$$

Setting $4A_0 t\,e^t + 4A_1 e^t = t\,e^t$, we see that $4A_0 = 1$ and $4A_1 = 0$, whereby $A_0 = \frac{1}{4}$ and $A_1 = 0$. Hence, a particular solution is $y_{P1}(t) = \frac{1}{4}t\,e^t$.

The forcing term $\cos(2t)$ has degree $d = 0$, characteristic $\mu + i\nu = i2$, and multiplicity $m = 0$. Because $m = 0$ and $m + d = 0$, we seek a particular solution of the form

$$y_{P2}(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$\begin{aligned} y'_{P2}(t) &= -2A\sin(2t) + 2B\cos(2t)\,, \\ y''_{P2}(t) &= -4A\cos(2t) - 4B\sin(2t)\,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}y_{P2}(t) &= y''_{P2}(t) - 2y'_{P2}(t) + 5y_{P2}(t) \\ &= \big(-4A\cos(2t) - 4B\sin(2t)\big) - 2\big(-2A\sin(2t) + 2B\cos(2t)\big) \\ &\quad + 5\big(A\cos(2t) + B\sin(2t)\big) \\ &= (A - 4B)\cos(2t) + (B + 4A)\sin(2t)\,. \end{aligned}$$

Setting $(A - 4B)\cos(2t) + (B + 4A)\sin(2t) = \cos(2t)$, we see that

$$A - 4B = 1\,, \qquad B + 4A = 0\,.$$

This system can be solved by any method you choose to find $A = \frac{1}{17}$ and $B = -\frac{4}{17}$, whereby a particular solution is

$$y_{P2}(t) = \tfrac{1}{17}\cos(2t) - \tfrac{4}{17}\sin(2t)\,.$$

Upon combining these two particular solutions with the general solution of the associated homogeneous problem found earlier we obtain the general solution

$$\begin{aligned} y &= y_H(t) + y_{P1}(t) + y_{P2}(t) \\ &= c_1 e^t\cos(2t) + c_2 e^t\sin(2t) + \tfrac{1}{4}t\,e^t + \tfrac{1}{17}\cos(2t) - \tfrac{4}{17}\sin(2t)\,. \end{aligned}$$

Solution (b). The equation is

$$\ddot{u} - 3\dot{u} - 10u = te^{-2t}\,.$$

This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 3z - 10 = (z - 5)(z + 2)\,.$$

This has the two real roots 5 and -2, which yields a general solution of the associated homogeneous problem

$$u_H(t) = c_1 e^{5t} + c_2 e^{-2t}\,.$$

The forcing $t\,e^{-2t}$ has characteristic form with degree $d = 1$, characteristic $\mu + i\nu = -2$, and multiplicity $m = 1$. Therefore a particular solution $u_P(t)$ can be found by either the method of Key Identity Evaluations or the method of Undetermined Coefficients.

Key Indentity Evaluations. Because $m = 1$ and $m + d = 2$, we will need the first and second derivative of the Key Identity, which are computed from the Key Identity as

$$\begin{aligned} \mathrm{L}(e^{zt}) &= (z^2 - 3z - 10)\,e^{zt}\,, \\ \mathrm{L}(t\,e^{zt}) &= (z^2 - 3z - 10)\,t\,e^{zt} + (2z - 3)\,e^{zt}\,, \\ \mathrm{L}(t^2 e^{zt}) &= (z^2 - 3z - 10)\,t^2 e^{zt} + 2(2z - 3)\,t\,e^{zt} + 2\,e^{zt}\,. \end{aligned}$$

Evaluate the last two of these at $z = -2$ to find

$$\mathrm{L}(t\,e^{-2t}) = -7e^{-2t}\,, \qquad \mathrm{L}(t^2 e^{-2t}) = -14t\,e^{-2t} + 2e^{-2t}\,.$$

By adding $\frac{2}{7}$ of the first to the second we get

$$\mathrm{L}\big(t^2 e^{-2t} + \tfrac{2}{7}t\,e^{-2t}\big) = -14t\,e^{-2t}\,.$$

Upon dividing this by -14 we obtain

$$\mathrm{L}\big(- \tfrac{1}{14} t^2 e^{-2t} - \tfrac{2}{98} t\, e^{-2t} \big) = t\, e^{-2t}\,,$$

whereby a particular solution is

$$u_P(t) = -\tfrac{1}{14} t^2 e^{-2t} - \tfrac{2}{98} t\, e^{-2t}\,.$$

Therefore a general solution is

$$u(t) = u_H(t) + u_P(t) = c_1 e^{5t} + c_2 e^{-2t} - \tfrac{1}{14} t^2 e^{-2t} - \tfrac{2}{98} t\, e^{-2t}\,.$$

Undetermined Coefficients. Because $\mu + i\nu = -2$ while $m = 1$ and $m + d = 2$, we seek a particular solution of the form

$$u_P(t) = A_0 t^2 e^{-2t} + A_1 t\, e^{-2t}\,.$$

Because

$$\begin{aligned} \dot{u}_P(t) &= -2A_0 t^2 e^{-2t} + (2A_0 - 2A_1) t\, e^{-2t} + A_1 e^{-2t}\,, \\ \ddot{u}_P(t) &= 4A_0 t^2 e^{-2t} + (-8A_0 + 4A_1) t\, e^{-2t} + (2A_0 - 4A_1) e^{-2t}\,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}u_P(t) &= \ddot{u}_P(t) - 3\dot{u}_P(t) - 10 u_P(t) \\ &= \big(4A_0 t^2 e^{-2t} + (-8A_0 + 4A_1) t\, e^{-2t} + (2A_0 - 4A_1) e^{-2t}\big) \\ &\quad - 3\big(- 2A_0 t^2 e^{-2t} + (2A_0 - 2A_1) t\, e^{-2t} + A_1 e^{-2t}\big) \\ &\quad - 10\big(A_0 t^2 e^{-2t} + A_1 t\, e^{-2t}\big) \\ &= -14 A_0 t\, e^{-2t} + (2A_0 - 7A_1) e^{-2t}\,. \end{aligned}$$

By setting $-14A_0 t\, e^{-2t} + (2A_0 - 7A_1)e^{-2t} = t\, e^{-2t}$ we see that

$$-14A_0 = 1\,, \qquad 2A_0 - 7A_1 = 0\,.$$

This system can be solved by any method you choose to find $A_0 = -\frac{1}{14}$ and $A_1 = -\frac{2}{98}$, whereby a particular solution is

$$u_P(t) = -\tfrac{1}{14} t^2 e^{-2t} - \tfrac{2}{98} t\, e^{-2t}\,.$$

Therefore a general solution is

$$u(t) = u_H(t) + u_P(t) = c_1 e^{5t} + c_2 e^{-2t} - \tfrac{1}{14} t^2 e^{-2t} - \tfrac{2}{98} t\, e^{-2t}\,.$$

Solution (c). The equation is

$$v'' + 9v = \cos(3t)\,.$$

This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 9 = z^2 + 3^2\,.$$

This has the conjugate pair of roots $\pm i3$, which yields a general solution of the associated homogeneous problem

$$v_H(t) = c_1 \cos(3t) + c_2 \sin(3t)\,.$$

The forcing $\cos(3t)$ has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = i3$, and multiplicity $m = 1$. Therefore a particular solution $v_P(t)$ can be found by either the method of Key Identity Evaluations, the Zero-Degree Formula, or the method of Undetermined Coefficients.

Key Indentity Evaluations. Because $m = 1$ and $m + d = 1$, we need the first derivative of the Key Identity, which is found as

$$\begin{aligned} \mathrm{L}(e^{zt}) &= (z^2 + 9)\, e^{zt} \,, \\ \mathrm{L}(t\, e^{zt}) &= (z^2 + 9)\, t\, e^{zt} + 2z\, e^{zt} \,. \end{aligned}$$

Evaluate the first derivative of the Key Identity at $z = i3$ to find that

$$\mathrm{L}(t\, e^{i3t}) = i6\, e^{i3t} \,.$$

Because $\cos(3t) = \mathrm{Re}(e^{i3t})$, upon dividing by $i6$ and taking the real part we see that a particular solution is

$$v_P(t) = \mathrm{Re}\big(\tfrac{1}{i6} t\, e^{i3t}\big) = \mathrm{Re}\big(\tfrac{1}{i6} t\,(\cos(3t) + i\sin(3t))\big) = \tfrac{1}{6} t\, \sin(3t) \,.$$

Therefore a general solution is

$$v(t) = v_H(t) + v_P(t) = c_1 \cos(3t) + c_2 \sin(3t) + \tfrac{1}{6} t\, \sin(3t) \,.$$

Zero-Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the form

$$f(t) = \alpha e^{\mu t} \cos(\nu t) + \beta e^{\mu t} \sin(\nu t) = e^{\mu t}\, \mathrm{Re}\big((\alpha - i\beta) e^{i\nu t}\big) \,,$$

this formula gives the particular solution

$$v_P(t) = t^m e^{\mu t}\, \mathrm{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)}\, e^{i\nu t} \right) \,.$$

For this problem $f(t) = \cos(3t) = \mathrm{Re}(e^{i3t})$ and $p(z) = z^2 + 9$, so that $\mu + i\nu = i3$, $\alpha - i\beta = 1$, $m = 1$, and $p'(z) = 2z$. whereby

$$\begin{aligned} v_P(t) = t\, \mathrm{Re}\left(\frac{e^{i3t}}{p'(i3)} \right) &= t\, \mathrm{Re}\left(\frac{\cos(3t) + i\sin(3t)}{i6} \right) \\ &= \tfrac{1}{6} t\, \mathrm{Re}(\sin(3t) - i\cos(3t)) = \tfrac{1}{6} t\, \sin(3t) \,. \end{aligned}$$

Therefore a general solution is

$$v(t) = v_H(t) + v_P(t) = c_1 \cos(3t) + c_2 \sin(3t) + \tfrac{1}{6} t\, \sin(3t) \,.$$

Undetermined Coefficients. Because $\mu + i\nu = i3$ and $m = m + d = 1$, we seek a particular solution in the form

$$v_P(t) = At\, \cos(3t) + Bt\, \sin(3t) \,.$$

Because

$$\begin{aligned} v_P'(t) &= -3At\, \sin(3t) + 3Bt\, \cos(3t) A\cos(3t) + B\sin(3t) \,, \\ v_P''(t) &= -9At\, \cos(3t) - 9Bt\, \sin(3t) - 6A\sin(3t) + 6B\cos(3t) \,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}v_P(t) &= v_P''(t) + 9v_P(t) \\ &= \big(-9At\,\cos(3t) - 9Bt\,\sin(3t) - 6A\sin(3t) + 6B\cos(3t)\big) \\ &\quad + 9\big(At\,\cos(3t) + Bt\,\sin(3t)\big) \\ &= -6A\sin(3t) + 6B\cos(3t)\,. \end{aligned}$$

By setting $-6A\sin(3t) + 6B\cos(3t) = \cos(3t)$ we see that $A = 0$ and $B = \frac{1}{6}$, whereby a particular solution is

$$v_P(t) = \tfrac{1}{6}t\,\sin(3t)\,.$$

Therefore a general solution is

$$v(t) = v_H(t) + v_P(t) = c_1\cos(3t) + c_2\sin(3t) + \tfrac{1}{6}t\,\sin(3t)\,.$$

Remark. Because of the simple form of this equation, if we had tried to solve it by either the Green Function or Variation of Parameters method then integrals that arise are not too difficult. However, it is not generally a good idea to use these methods for such problems because evaluating the integrals that arise often involve much more work that the methods shown above.

Solution (d). The equation is

$$w'''' + 13w'' + 36w = 9\sin(t)\,.$$

This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^4 + 13z^2 + 36 = (z^2+4)(z^2+9) = (z^2+2^2)(z^2+3^2)\,.$$

This has the two conjugate pair of roots $\pm i2$ and $\pm i3$, which yields a general solution of the associated homogeneous problem

$$w_H(t) = c_1\cos(2t) + c_2\sin(2t) + c_3\cos(3t) + c_4\sin(3t)\,.$$

The forcing $9\sin(t)$ has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = i$, and multiplicity $m = 0$. Therefore a particular solution $w_P(t)$ can be found by either the method of Key Identity Evaluations, the Zero-Degree Formula, or the method of Undetermined Coefficients.

Key Identity Evaluations. Because $m = 0$ and $m + d = 0$, we need only the Key Identity, which is

$$\mathrm{L}(e^{zt}) = (z^4 + 13z^2 + 36)\,e^{zt}\,.$$

Evaluate this at $z = i$ to obtain

$$\mathrm{L}(e^{it}) = (i^4 + 13\cdot i^2 + 36)e^{it} = 24e^{it}\,.$$

Because $9\sin(t) = \mathrm{Re}(-i9e^{it})$, upon multiplying by $-i9/24$ and taking the real part we see that a particular solution is

$$w_P(t) = \mathrm{Re}\left(\frac{-i9}{24}e^{it}\right) = \tfrac{3}{8}\,\mathrm{Re}(\sin(t) - i\cos(t)) = \tfrac{3}{8}\sin(t)\,.$$

Therefore a general solution is

$$\begin{aligned} w(t) &= w_H(t) + w_P(t) \\ &= c_1 \cos(2t) + c_2 \sin(2t) + c_3 \cos(3t) + c_4 \sin(3t) + \tfrac{3}{8} \sin(t) \,. \end{aligned}$$

Zero-Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the form

$$f(t) = \alpha e^{\mu t} \cos(\nu t) + \beta e^{\mu t} \sin(\nu t) = e^{\mu t} \operatorname{Re}\big((\alpha - i\beta) e^{i\nu t}\big) \,,$$

this formula gives the particular solution

$$w_P(t) = t^m e^{\mu t} \operatorname{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)} \, e^{i\nu t} \right) \,.$$

For this problem $f(t) = 9\sin(t)$ and $p(z) = z^4 + 13z^2 + 36$, so that $\mu + i\nu = i$, $\alpha - i\beta = -i9$, and $m = 0$, whereby

$$\begin{aligned} w_P(t) &= \operatorname{Re}\left(\frac{-i9}{p(i)} \, e^{it} \right) = \operatorname{Re}\left(\frac{-i9}{i^4 + 13 \cdot i^2 + 36} \, e^{it} \right) \\ &= \operatorname{Re}\left(\frac{-i9}{24} \, \big(\cos(t) + i \sin(t) \big) \right) = \tfrac{3}{8} \operatorname{Re}(\sin(t) - i \cos(t)) = \tfrac{3}{8} \sin(t) \,. \end{aligned}$$

Therefore a general solution is

$$\begin{aligned} w(t) &= w_H(t) + w_P(t) \\ &= c_1 \cos(2t) + c_2 \sin(2t) + c_3 \cos(3t) + c_4 \sin(3t) + \tfrac{3}{8} \sin(t) \,. \end{aligned}$$

Undetermined Coefficients. Because $\mu + i\nu = i$ and $m = m + d = 0$, we seek a particular solution in the form

$$w_P(t) = A \cos(t) + B \sin(t) \,.$$

Because

$$\begin{aligned} w_P'(t) &= -A \sin(t) + B \cos(t) \,, \\ w_P''(t) &= -A \cos(t) - B \sin(t) \,, \\ w_P'''(t) &= A \sin(t) - B \cos(t) \,, \\ w_P''''(t) &= A \cos(t) + B \sin(t) \,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L} w_P(t) &= w_P''''(t) + 13 w_P''(t) + 36 w_P(t) \\ &= \big(A \cos(t) + B \sin(t) \big) + 13 \big(- A \cos(t) - B \sin(t) \big) \\ &\quad + 36 \big(A \cos(t) + B \sin(t) \big) \\ &= 24A \cos(t) + 24B \sin(t) \,. \end{aligned}$$

By setting $24A\cos(t) + 24B\sin(t) = 9\sin(t)$ we see that $A = 0$ and $24B = 9$, whereby a particular solution is

$$w_P(t) = \tfrac{3}{8} \sin(t) \,.$$

Therefore a general solution is

$$\begin{aligned} w(t) &= w_H(t) + w_P(t) \\ &= c_1 \cos(2t) + c_2 \sin(2t) + c_3 \cos(3t) + c_4 \sin(3t) + \tfrac{3}{8} \sin(t) \,. \end{aligned}$$

(10) Solve the following initial-value problems.

(a) $w'' + 4w' + 20w = 5e^{2t}$, $\qquad w(0) = 3\,, \quad w'(0) = -7.$

(b) $y'' - 4y' + 4y = \dfrac{e^{2t}}{3+t}$, $\qquad y(0) = 0\,, \quad y'(0) = 5.$

Evaluate any definite integrals that arise.

Solution (a). This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 4z + 20 = (z+2)^2 + 16 = (z+2)^2 + 4^2\,.$$

This has the conjugate pair of roots $-2 \pm i4$, which yields a general solution of the associated homogeneous problem

$$w_H(t) = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t)\,.$$

The forcing $5e^{2t}$ has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = 2$, and multiplicity $m = 0$. Therefore a particular solution $w_P(t)$ can be found by either the method of Key Identity Evaluations, the Zero-Degree Formula, or the method of Undetermined Coefficients.

Key Indentity Evaluations. Because $m = 0$ and $m + d = 0$, we only need the Key Identity,

$$\mathrm{L}(e^{zt}) = (z^2 + 4z + 20)\, e^{zt}\,.$$

Evaluate this at $z = 2$ to find that

$$\mathrm{L}(e^{2t}) = (4 + 8 + 20)e^{2t} = 32e^{2t}\,.$$

Upon multiplying this by $\frac{5}{32}$ we see that a particular solution is

$$w_P(t) = \tfrac{5}{32}e^{2t}\,.$$

Zero-Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big)\,,$$

this formula gives the particular solution

$$w_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)}\, e^{i\nu t}\right)\,.$$

For this problem $f(t) = 5e^{2t}$ and $p(z) = z^2 + 4z + 20$, so that $\mu + i\nu = 2$, $\alpha - i\beta = 5$, and $m = 0$, whereby

$$w_P(t) = e^{2t}\frac{5}{p(2)} = \tfrac{5}{32}e^{2t}\,.$$

Undetermined Coefficients. Because $\mu + i\nu = 2$ and $m = m + d = 0$, we seek a particular solution in the form

$$w_P(t) = Ae^{2t}\,.$$

Because

$$w_P'(t) = 2Ae^{2t}\,, \qquad w_P''(t) = 4Ae^{2t}\,,$$

we see that

$$\begin{aligned} \mathrm{L}w_P(t) &= w_P''(t) + 4w_P'(t) + 20w_P(t) \\ &= 4Ae^{2t} + 4(2Ae^{2t}) + 20Ae^{2t} = 32Ae^{2t} \,. \end{aligned}$$

By setting $32Ae^{2t} = 5e^{2t}$, we see that $A = \frac{5}{32}$, whereby a particular solution is

$$w_P(t) = \tfrac{5}{32}e^{2t} \,.$$

Solving the Initial-Value Problem. By either method we find that a general solution is

$$w(t) = w_H(t) + w_P(t) = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t) + \tfrac{5}{32}e^{2t} \,.$$

Because

$$\begin{aligned} w'(t) = &-2c_1 e^{-2t}\cos(4t) - 4c_1 e^{-2t}\sin(4t) \\ &- 2c_2 e^{-2t}\sin(4t) + 4c_2 e^{-2t}\cos(4t) + \tfrac{5}{16}e^{2t} \,, \end{aligned}$$

the initial conditions yield

$$3 = w(0) = c_1 + \tfrac{5}{32} \,, \qquad -7 = w'(0) = -2c_1 + 4c_2 + \tfrac{5}{16} \,.$$

Upon solving this system we find that $c_1 = \frac{91}{32}$ and $c_2 = -\frac{13}{32}$, whereby the solution of the initial-value problem is

$$w(t) = \tfrac{91}{32}e^{-2t}\cos(4t) - \tfrac{13}{32}e^{-2t}\sin(4t) + \tfrac{5}{32}e^{2t} \,.$$

Solution (b). The initial-value problem is

$$y'' - 4y' + 4y = \frac{e^{2t}}{3+t} \,, \qquad y(0) = 0 \,, \quad y'(0) = 5 \,.$$

This is a constant coefficient, nonhomogeneous, linear equation in normal form. Its characteristic polynomial is

$$p(z) = z^2 - 4z + 4 = (z-2)^2 \,.$$

This has the double real root 2, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^{2t} + c_2 t\, e^{2t} \,.$$

The forcing $e^{2t}/(3+t)$ does not have characteristic form. Therefore a particular solution $y_P(t)$ *cannot* be found by either the method of Key Identity Evaluations or the method of Undetermined Coefficients. Moreover, the forcing does not have a form found in the table of Laplace transforms, so that method cannot be used either. Rather, we must use either the Green Function or the Variation of Parameters method.

Green Function. The associated Green function $g(t)$ satisfies

$$g'' - 4g' + 4g = 0 \,, \qquad g(0) = 0 \,, \quad g'(0) = 1 \,.$$

Because its characteristic polynomial is $p(z) = z^2 - 4z + 4 = (z-2)^2$, a general solution of this equation is

$$g(t) = c_1 e^{2t} + c_2 t\, e^{2t} \,.$$

Because $0 = g(0) = c_1$, we see that $g(t) = c_2 t\, e^{2t}$. Then

$$g'(t) = c_2 e^{2t} + 2c_2 t\, e^{2t}\,.$$

Because $1 = g'(0) = c_2$, the Green function is $g(t) = t\, e^{2t}$.

Alternatively, because its characteristic polynomial is $p(z) = z^2 - 4z + 4 = (z-2)^2$, its Green function is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right](t) = t\, e^{2t}\,.$$

Referring to the Table of Laplace Transforms on the last page, here we have used item 1 with $n = 1$ and $a = 2$.

The particular solution $y_P(t)$ that satisfies $y_P(0) = y_P'(0) = 0$ is given by

$$\begin{aligned} y_P(t) &= \int_0^t g(t-s)\frac{e^{2s}}{3+s}\,\mathrm{d}s = \int_0^t (t-s)e^{2t-2s}\frac{e^{2s}}{3+s}\,\mathrm{d}s \\ &= e^{2t}\int_0^t \frac{t-s}{3+s}\,\mathrm{d}s = e^{2t}t\int_0^t \frac{1}{3+s}\,\mathrm{d}s - e^{2t}\int_0^t \frac{s}{3+s}\,\mathrm{d}s\,. \end{aligned}$$

Because

$$\begin{aligned} &\int_0^t \frac{1}{3+s}\,\mathrm{d}s = \log(3+s)\Big|_0^t = \log(3+t) - \log(3) = \log\left(\frac{3+t}{3}\right), \\ &\int_0^t \frac{s}{3+s}\,\mathrm{d}s = \int_0^t 1 - \frac{3}{3+s}\,\mathrm{d}s = t - 3\log\left(\frac{3+t}{3}\right), \end{aligned}$$

we find that

$$y_P(t) = e^{2t}t\log(1+\tfrac{1}{3}t) - e^{2t}\big(t - 3\log(1+\tfrac{1}{3}t)\big)\,.$$

Therefore a general solution of the equation is

$$y(t) = c_1 e^{2t} + c_2 t\, e^{2t} + y_P(t)\,.$$

Because

$$y'(t) = 2c_1 e^{2t} + 2c_2 t\, e^{2t} + c_2 e^{2t} + y_P'(t)\,,$$

and because $y_P(0) = y_P'(0) = 0$, the initial conditions imply

$$0 = y(0) = c_1\,, \qquad 5 = y'(0) = 2c_1 + c_2\,.$$

We find that $c_1 = 0$ and $c_2 = 5$, whereby the solution of the initial-value problem is

$$y(t) = 5t\, e^{2t} + e^{2t}t\log(1+\tfrac{1}{3}t) - e^{2t}\big(t - 3\log(1+\tfrac{1}{3}t)\big)\,.$$

Variation of Parameters. The equation is already in normal form. Therefore we *seek* a particular solution of the form

$$y_P(t) = e^{2t}u_1(t) + t\, e^{2t}u_2(t)\,,$$

such that

$$\begin{aligned} e^{2t}u_1'(t) + t\, e^{2t}u_2'(t) &= 0\,, \\ 2e^{2t}u_1'(t) + (2t\, e^{2t} + e^{2t})u_2'(t) &= \frac{e^{2t}}{3+t}\,. \end{aligned}$$

This system can be solved to find that

$$u_1'(t) = -\frac{t}{3+t}\,, \qquad u_2'(t) = \frac{1}{3+t}\,.$$

These can be integrated to obtain

$$\begin{aligned} u_1(t) &= -\int \frac{t}{3+t}\,\mathrm{d}t = -\int 1 - \frac{3}{3+t}\,\mathrm{d}t = -t + 3\log(3+t) + c_1\,, \\ u_2(t) &= \int \frac{1}{3+t}\,\mathrm{d}t = \log(3+t) + c_2\,, \end{aligned}$$

whereby a general solution is

$$y(t) = c_1 e^{2t} - e^{2t}\big(t - 3\log(3+t)\big) + c_2 t\, e^{2t} + t\, e^{2t}\log(3+t)\,.$$

Because

$$\begin{aligned} y'(t) = 2c_1e^{2t} &- 2e^{2t}\big(t - 3\log(3+t)\big) - e^{2t}\left(1 - \frac{3}{3+t}\right) \\ &+ 2c_2 t\, e^{2t} + c_2 e^{2t} + 2t\, e^{2t}\log(3+t) + e^{2t}\log(3+t) + t\, e^{2t}\frac{1}{3+t}\,, \end{aligned}$$

the initial conditions imply that

$$\begin{aligned} 0 &= y(0) = c_1 + 3\log(3)\,, \\ 5 &= y'(0) = 2c_1 + 6\log(3) + c_2 + \log(3)\,. \end{aligned}$$

We can solve this system to find that $c_1 = -3\log(3)$ and $c_2 = 5 - \log(3)$. Therefore the solution of the initial-value problem is

$$y(t) = -3\log(3)e^{2t} - e^{2t}\big(t - 3\log(3+t)\big) + \big(5 - \log(3)\big)t\, e^{2t} + t\, e^{2t}\log(3+t)\,.$$

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(11) Given that $y_1(t) = t$ and $y_2(t) = t^{-2}$ solve the associated homogeneous equation, find a general solution of

$$t^2 y'' + 2t\, y' - 2y = \frac{3}{t^2} + 5t\,, \qquad \text{for } t > 0\,.$$

Solution. This is a *variable coefficient, nonhomogeneous,* linear equation. We are given that $y_1(t) = t$ and $y_2(t) = t^{-2}$ are solutions of the associated homogeneous equation. Their Wronskian is

$$\mathrm{Wr}[t, t^{-2}] = \det\begin{pmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{pmatrix} = -2t^{-2} - t^{-2} = -3t^{-2}\,.$$

Because $\mathrm{Wr}[t, t^{-2}] \neq 0$ for $t > 0$, $y_1(t) = t$ and $y_2(t) = t^{-2}$ are a fundamental set of solutions for the associated homogeneous equation, and a general solution of that equation is

$$y_H(t) = c_1 t + c_2 t^{-2}\,.$$

To find a general solution of the given nonhomogeneous equation we need to find a particular solution $y_P(t)$ of that equation. Because this equation has variable coefficients, we will use either the General Green Function or the Variation of Parameters method. To use either method we must put the equation into its normal form

$$y'' + \frac{2}{t}\, y' - \frac{2}{t^2} y = \frac{3}{t^4} + \frac{5}{t} \,, \qquad \text{for } t > 0 \,.$$

General Green Function. The Green function is given

$$G(t,s) = \frac{1}{\mathrm{Wr}[s, s^{-2}]} \det\begin{pmatrix} s & s^{-2} \\ t & t^{-2} \end{pmatrix} = -\tfrac{1}{3} s^2 \big(t^{-2}s - t\, s^{-2}\big) = \tfrac{1}{3}\big(t - t^{-2}s^3\big) \,.$$

Then the particular solution that satisfies $y(1) = y'(1) = 0$ is given by

$$\begin{aligned} y_P(t) &= \int_1^t G(t,s)\, f(s)\, ds = \tfrac{1}{3} \int_1^t \big(t - t^{-2}s^3\big) \left(\frac{3}{s^4} + \frac{5}{s}\right) \mathrm{d}s \\ &= t \int_1^t \frac{1}{s^4}\, \mathrm{d}s - t^{-2} \int_1^t \frac{1}{s}\, \mathrm{d}s + \tfrac{5}{3} t \int_1^t \frac{1}{s}\, \mathrm{d}s - \tfrac{5}{3} t^{-2} \int_1^t s^2\, \mathrm{d}s \\ &= \tfrac{1}{3} t\big(1 - t^{-3}\big) - t^{-2}\log(t) + \tfrac{5}{3} t \log(t) - \tfrac{5}{9} t^{-2}(t^3 - 1) \,. \end{aligned}$$

Because the first and last terms above are solutions of the associated homogeneous equation, a general solution can be expressed as

$$y(t) = c_1 t + c_2 t^{-2} - t^{-2}\log(t) + \tfrac{5}{3} t \log(t) \,.$$

Variation of Parameters. Seek a general solution in the form

$$y(t) = t\, u_1(t) + t^{-2} u_2(t) \,,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$\begin{aligned} t\, u_1'(t) + t^{-2} u_2'(t) &= 0 \,, \\ 1\, u_1'(t) - 2t^{-3} u_2'(t) &= 3t^{-4} + 5t^{-1} \,. \end{aligned}$$

The solution of this system is

$$u_1'(t) = \tfrac{1}{3}\big(3t^{-4} + 5t^{-1}\big) \,, \qquad u_2'(t) = -\tfrac{1}{3} t^3 \big(3t^{-4} + 5t^{-1}\big) \,.$$

Upon integrating these equations we find that

$$u_1(t) = c_1 - \tfrac{1}{3} t^{-3} + \tfrac{5}{3}\log(t) \,, \qquad u_2(t) = c_2 - \log(t) - \tfrac{5}{9} t^3 \,.$$

Therefore a general solution of the nonhomogeneous linear equation is

$$y(t) = c_1 t + c_2 t^{-2} - \tfrac{1}{3} t^{-2} + \tfrac{5}{3} t \log(t) - t^{-2}\log(t) - \tfrac{5}{9} t \,.$$

---

(12) Given that t^2 and $t^2 \log(t)$ solve the associated homogeneous differential equation, consider the initial-value problem

$$t^2 x'' - 3t\, x' + 4x = t^2 \log(t) \,, \qquad x(1) = 0 \,, \quad x'(1) = 0 \,.$$

(a) Give the interval of definition of its solution.
(b) Compute $\mathrm{Wr}[t^2, t^2 \log(t)]$.
(c) Find $x(t)$. Evaluate any definite integrals that arise.

Solution (a). This is a *nonhomogeneous, linear*, second-order initial-value problem with *variable coefficients*. Its normal form is

$$x'' - \frac{3}{t}\,x' + \frac{4}{t^2}\,x = \log(t)\,, \qquad x(1) = 0\,, \quad x'(1) = 0\,.$$

We see that

- the coefficients of x' and x are undefined at $t = 0$ and are continuous elsewhere,
- the forcing is undefined for $t \leq 0$ and is continuous elsewhere.

Because the initial time is $t = 1$ we conclude that the interval of definition for the solution will by $(0, \infty)$ because

- the initial time $t = 1$ is in $(0, \infty)$;
- the forcing and bothof the coefficients are continuous over $(0, \infty)$;
- the forcing and each of the coefficients is undefined at $t = 0$.

Solution (b). We know that t^2 and $t^2 \log(t)$ solve the associated homogeneous equation. Their Wronskian is

$$\begin{aligned}\mathrm{Wr}[t^2, t^2\log(t)] &= \det\begin{pmatrix} t^2 & t^2\log(t) \\ 2t & 2t\log(t) + t\end{pmatrix} \\ &= t^2 \cdot \big(2t\log(t) + t\big) - 2t \cdot t^2\log(t) = t^3\,.\end{aligned}$$

Because $\mathrm{Wr}[t^2, t^3] \neq 0$ for $t > 0$, we see that t^2 and $t^2 \log(t)$ are a fundamental set of solutions for the associated homogeneous equation.

Solution (c). Because this is an initial-value problem, the General Green Function method is the quickest route to the answer. However, the problem can also be solved by the Variation of Parameters method. To use either method we must put the initial-value problem into its normal form

$$x'' - \frac{3}{t}\,x' + \frac{4}{t^2}\,x = \log(t)\,, \qquad x(1) = 0\,, \quad x'(1) = 0\,.$$

General Green Function. The Green function is given

$$\begin{aligned}G(t,s) &= \frac{1}{\mathrm{Wr}[s^2, s^2\log(s)]}\det\begin{pmatrix} s^2 & s^2\log(s) \\ t^2 & t^2\log(t)\end{pmatrix} \\ &= \frac{1}{s^3}\big(t^2\log(t)s^2 - t^2 s^2\log(s)\big) = t^2\log(t)\frac{1}{s} - t^2\frac{\log(s)}{s}\,.\end{aligned}$$

Then the solution that satisfies $x(1) = x'(1) = 0$ is given by

$$\begin{aligned}x(t) = \int_1^t G(t,s)\,f(s)\,ds &= \int_1^t \left(t^2\log(t)\frac{1}{s} - t^2\frac{\log(s)}{s}\right)\log(s)\,\mathrm{d}s \\ &= t^2\log(t)\int_1^t \frac{\log(s)}{s}\,\mathrm{d}s - t^2\int_1^t \frac{\big(\log(s)\big)^2}{s}\,\mathrm{d}s \\ &= t^2\log(t)\cdot\tfrac{1}{2}\big(\log(t)\big)^2 - t^2\cdot\tfrac{1}{2}\big(\log(t)\big)^2 = \tfrac{1}{6}t^2\big(\log(t)\big)^3\,.\end{aligned}$$

This is the solution of the initial-value problem.

Variation of Parameters. Because t^2 and $t^2 \log(t)$ are a fundamental set of solutions for the associated homogeneous equation, a general solution of that equation is

$$x_H(t) = c_1 t^2 + c_2 t^2 \log(t) \,.$$

Therefore we seek a general solution of the nonhomogeneous equation in the form

$$x(t) = t^2 u_1(t) + t^2 \log(t) u_2(t) \,,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$\begin{aligned} t^2 u_1'(t) \qquad\quad &+ t^2 \log(t) u_2'(t) = 0 \,, \\ 2t\, u_1'(t) + \big(2t \log(t) + t\big)\, u_2'(t) &= \log(t) \,. \end{aligned}$$

The solution of this system is

$$u_1'(t) = -\frac{\big(\log(t)\big)^2}{t} \,, \qquad u_2'(t) = \frac{\log(t)}{t} \,.$$

Upon integrating these equations we obtain

$$u_1(t) = -\tfrac{1}{3}\big(\log(t)\big)^3 + c_1 \,, \qquad u_2(t) = \tfrac{1}{2}\big(\log(t)\big)^2 + c_2 \,.$$

Therefore a general solution of the nonhomogeneous linear equation is

$$x(t) = c_1 t^2 + c_2 t^2 \log(t) + \tfrac{1}{6} t^2 \big(\log(t)\big)^3 \,.$$

We must find the values of c_1 and c_2 for which this general solution satisfies the initial conditions. Because

$$x(1) = c_1 1^2 + c_2 1^2 \log(1) + \tfrac{1}{6} 1^2 \big(\log(1)\big)^3 = c_1 \,,$$

the initial condition $x(1) = 0$ implies that $c_1 = 0$. Because

$$x'(t) = c_2\big(2t \log(t) + t\big) + \tfrac{1}{3} t \,\big(\log(t)\big)^3 + \tfrac{1}{2} t \,\big(\log(t)\big)^2 \,,$$

we have

$$x'(1) = c_2\big(2 \log(1) + 1\big) + \tfrac{1}{3} \,\big(\log(1)\big)^3 + \tfrac{1}{2} \,\big(\log(1)\big)^2 = c_2 \,,$$

whereby the initial condition $x'(1) = 0$ implies that $c_2 = 0$. Therefore the solution of the intial-value problem is

$$x(t) = \tfrac{1}{6} t^2 \big(\log(t)\big)^3 \,.$$

---

(13) Give an explicit general solution of the equation

$$\ddot{h} + 2\dot{h} + 5h = 0 \,.$$

Sketch a typical solution for $t \geq 0$. If this equation governs a spring-mass system, is the system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

Solution. This is a constant coefficient, homogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + 2z + 5 = (z+1)^2 + 2^2 \,.$$

This has the conjugate pair of roots $-1 \pm i2$, which yields a general solution

$$h(t) = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t) \,.$$

When $c_1^2 + c_2^2 > 0$ this can be put into the amplitute-phase form

$$h(t) = Ae^{-t}\cos(2t - \delta)\,,$$

where $A > 0$ and $0 \le \delta < 2\pi$ are determined from c_1 and c_2 by

$$A = \sqrt{c_1^2 + c_2^2}\,, \qquad \cos(\delta) = \frac{c_1}{A}\,, \qquad \sin(\delta) = \frac{c_2}{A}\,.$$

In other words, (A, δ) are the polar coordinates for the point in the plane whose Cartesian coordinates are (c_1, c_2). The sketch should show a decaying oscillation with amplitude Ae^{-t} and damped period $\frac{2\pi}{2} = \pi$. A sketch might be given during the review session. The equation governs an *under damped* spring-mass system because its characteristic polynomial has a conjugate pair of roots with negative real part.

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(14) When a mass of 2 kilograms is hung vertically from a spring, it stretches the spring 0.5 meters. (Gravitational acceleration is 9.8 m/sec^2.) At $t = 0$ the mass is set in motion from 0.3 meters below its rest (equilibrium) position with a upward velocity of 2 m/sec. It is acted upon by an external force of $2\cos(5t)$. Neglect damping and assume that the spring force is proportional to its displacement. Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (Do not solve this initial-value problem; just write it down!)

Solution. Let $h(t)$ be the displacement (in meters) of the mass from its equilibrium (rest) position at time t (in seconds), with *upward displacements being positive.* The governing initial-value problem then has the form

$$m\ddot{h} + kh = 2\cos(5t)\,, \qquad h(0) = -.3\,, \quad h'(0) = 2\,,$$

where m is the mass and k is the spring constant. The problem says that $m = 2$ kilograms. The spring constant is obtained by balancing the weight of the mass (mg $= 2 \cdot 9.8$ Newtons) with the force applied by the spring when it is stetched .5 m. This gives $k\,.5 = 2 \cdot 9.8$, or

$$k = \frac{2 \cdot 9.8}{.5} = 4 \cdot 9.8 \quad \text{Newtons/m}\,.$$

Therefore the governing initial-value problem is

$$2\ddot{h} + 4 \cdot 9.8h = 2\cos(5t)\,, \qquad h(0) = -.3\,, \quad h'(0) = 2\,.$$

Had you chosen *downward displacements to be positive* then the sign of the initial data would change! You should make your convention clear!

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(15) Find the Laplace transform $Y(s)$ of the solution $y(t)$ to the initial-value problem

$$y'' + 4y' + 8y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 4\,.$$

where

$$f(t) = \begin{cases} 4 & \text{for } 0 \le t < 2\,, \\ t^2 & \text{for } 2 \le t\,. \end{cases}$$

You may refer to the table of Laplace transforms on the last page. (Do not take the inverse Laplace transform to find $y(t)$; just solve for $Y(s)$!)

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 8\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned}\mathcal{L}[y](s) &= Y(s)\,,\\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = sY(s) - 2\,,\\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - 2s - 4\,.\end{aligned}$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned}f(t) &= \big(1 - u(t-2)\big)4 + u(t-2)t^2 = 4 - u(t-2)4 + u(t-2)t^2\\ &= 4 + u(t-2)(t^2-4) = 4 + u(t-2)j(t-2)\,,\end{aligned}$$

where by the shifty step method

$$j(t) = (t+2)^2 - 4 = t^2 + 4t\,.$$

Referring to the table of Laplace transforms, item 9 with $c = 2$ and $j(t) = t^2 + 4t$, and item 1 with $a = 0$ and $n = 1$, and with $a = 0$ and $n = 2$ then show that

$$\begin{aligned}\mathcal{L}[f](s) &= 4\mathcal{L}[1](s) + \mathcal{L}\big[u(t-2)j(t-2)\big](s)\\ &= 4\mathcal{L}[1](s) + e^{-2s}\mathcal{L}\big[j(t)\big](s)\\ &= 4\mathcal{L}[1](s) + e^{-2s}\mathcal{L}[4t+t^2](s)\\ &= 4\mathcal{L}[1](s) + 4e^{-2s}\mathcal{L}[t](s) + e^{-2s}\mathcal{L}[t^2](s)\\ &= 4\frac{1}{s} + 4e^{-2s}\frac{1}{s^2} + e^{-2s}\frac{2}{s^3}\,.\end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 2s - 4\big) + 4\big(sY(s) - 2\big) + 8Y(s) = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}\,,$$

which becomes

$$(s^2 + 4s + 8)Y(s) - 2s - 12 = \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2+4s+8}\left(2s + 12 + \frac{4}{s} + e^{-2s}\frac{4}{s^2} + e^{-2s}\frac{2}{s^3}\right).$$

---

(16) Let $x(t)$ be the solution of the initial-value problem

$$x'' + 10x' + 29x = f(t)\,, \qquad x(0) = 3\,, \quad x'(0) = -7\,,$$

where the forcing $f(t)$ is given by

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 1\,,\\ e^{1-t} & \text{for } 1 \le t < \infty\,.\end{cases}$$

(a) Find the Laplace transform $F(s)$ of the forcing $f(t)$.
(b) Find the Laplace transform $X(s)$ of the solution $x(t)$.
(DO NOT take the inverse Laplace transform to find $x(t)$; just solve for $X(s)$!)

You may refer to the table of Laplace transforms on the last page.

Solution (a). First write $f(t)$ as

$$f(t) = t^2 + u(t-1)\big(e^{1-t} - t^2\big) = t^2 + u(t-1)j(t-1)\,,$$

where by the Shifty Step Method

$$j(t) = e^{1-(t+1)} - (t+1)^2 = e^{-t} - t^2 - 2t - 1\,.$$

Referring to the Table of Laplace Transforms, item 1 with $n = 0$ and $a = -1$, with $n = 2$ and $a = 0$, with $n = 1$ and $a = 0$, and with $n = 0$ and $a = 0$ gives

$$\begin{aligned} J(s) = \mathcal{L}[j](s) &= \mathcal{L}[e^{-t}](s) - \mathcal{L}[t^2](s) - 2\mathcal{L}[t](s) - \mathcal{L}[1](s) \\ &= \frac{1}{s+1} - \frac{2}{s^3} - \frac{2}{s^2} - \frac{1}{s}\,. \end{aligned}$$

Then item 1 with $n = 2$ and $a = 0$ and item 9 with $c = 1$ gives

$$\begin{aligned} F(s) = \mathcal{L}[f](s) &= \mathcal{L}[t^2](s) + \mathcal{L}[u(t-1)j(t-1)](s) = \frac{2}{s^3} + e^{-s}J(s) \\ &= \frac{2}{s^3} + e^{-s}\left[\frac{1}{s+1} - \frac{2}{s^3} - \frac{2}{s^2} - \frac{1}{s}\right]. \end{aligned}$$

Solution (b). The initial-value problem

$$x'' + 10x' + 29x = f(t)\,, \qquad x(0) = 3\,, \quad x'(0) = -7\,,$$

has Laplace transform

$$\mathcal{L}[x''](s) + 10\mathcal{L}[x'](s) + 29\mathcal{L}[x](s) = F(s)\,,$$

where $F(s) = \mathcal{L}[f](s)$ was computed in part (a) and

$$\begin{aligned} \mathcal{L}[x](s) &= X(s)\,, \\ \mathcal{L}[x'](s) &= s\,\mathcal{L}[x](s) - x(0) = s\,X(s) - 3\,, \\ \mathcal{L}[x''](s) &= s\,\mathcal{L}[x'](s) - x'(0) = s^2X(s) - 3s + 7\,. \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2X(s) - 3s + 7\big) + 10\big(sX(s) - 3\big) + 29X(s) = F(s)\,,$$

which becomes

$$(s^2 + 10s + 29)X(s) - 3s - 23 = F(s)\,.$$

Therefore $X(s)$ is

$$X(s) = \frac{1}{s^2 + 10s + 29}\left(\frac{2}{s^3} + e^{-s}\left[\frac{1}{s+1} - \frac{2}{s^3} - \frac{2}{s^2} - \frac{1}{s}\right] + 3s + 23\right).$$

(17) Find the function $y(t)$ whose Laplace transform $Y(s)$ is given by

$$\text{(a)}\quad Y(s) = \frac{e^{-3s}4}{s^2 - 6s + 5}\,, \qquad\qquad \text{(b)}\quad Y(s) = \frac{e^{-2s}s}{s^2 + 4s + 8}\,.$$

You may refer to the table of Laplace transforms on the last page.

Solution (a). The denominator factors as $(s-5)(s-1)$, so we have the partial fraction identity

$$\frac{4}{s^2 - 6s + 5} = \frac{4}{(s-5)(s-1)} = \frac{1}{s-5} - \frac{1}{s-1}\,.$$

Referring to the table of Laplace transforms, item 1 with $a = 5$ and $n = 0$ and with $a = 1$ and $n = 0$ gives

$$\mathcal{L}^{-1}\left[\frac{1}{s-5}\right](t) = e^{5t}\,, \qquad \mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) = e^{t}\,,$$

whereby

$$\mathcal{L}^{-1}\left[\frac{4}{s^2 - 6s + 5}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s-5} - \frac{1}{s-1}\right] = e^{5t} - e^{t}\,.$$

It follows from line 9 with $c = 3$ and $j(t) = e^{5t} - e^{t}$ that

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)](t) &= \mathcal{L}^{-1}\left[\frac{e^{-3s}4}{s^2 - 6s + 5}\right](t) \\ &= u(t-3)\mathcal{L}^{-1}\left[\frac{4}{s^2 - 6s + 5}\right](t-3) = u(t-3)\big(e^{5(t-3)} - e^{t-3}\big)\,. \end{aligned}$$

Solution (b). The denominator does not have real factors. The partial fraction identity is

$$\frac{s}{s^2 + 4s + 8} = \frac{s}{(s+2)^2 + 4} = \frac{s+2}{(s+2)^2 + 2^2} - \frac{2}{(s+2)^2 + 2^2}\,.$$

Referring to the table of Laplace transforms, items 2 and 3 with $a = -2$ and $b = 2$ give

$$\mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2 + 2^2}\right](t) = e^{-2t}\cos(2t)\,, \quad \mathcal{L}^{-1}\left[\frac{2}{(s+2)^2 + 2^2}\right](t) = e^{-2t}\sin(2t)\,,$$

whereby

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{s}{s^2 + 4s + 8}\right](t) &= \mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2 + 2^2}\right](t) - \mathcal{L}^{-1}\left[\frac{2}{(s+2)^2 + 2^2}\right](t) \\ &= e^{-2t}\big(\cos(2t) - \sin(2t)\big)\,. \end{aligned}$$

It follows from line 9 with $c = 2$ and $j(t) = e^{-2t}\big(\cos(2t) - \sin(2t)\big)$ that

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y(s)](t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}s}{s^2 + 4s + 8}\right](t) = u(t-2)\mathcal{L}^{-1}\left[\frac{s}{s^2 + 4s + 8}\right](t-2) \\ &= u(t-2)e^{-2(t-2)}\big(\cos(2(t-2)) - \sin(2(t-2))\big)\,. \end{aligned}$$

(18) Consider the real vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^3 \\ 3+t^4 \end{pmatrix}$.

(a) Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the linear system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.

(c) Give a general solution to the system you found in part (b).

Solution (a). The Wronskian is given by

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 1 & t^3 \\ t & 3+t^4 \end{pmatrix} = 1 \cdot (3+t^4) - t \cdot t^3 = 3 + t^4 - t^4 = 3\,.$$

Solution (b). Set

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & t^3 \\ t & 3+t^4 \end{pmatrix}.$$

If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ each satisfy $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ then $\mathbf{\Psi}(t)$ must satisfy

$$\mathbf{\Psi}'(t) = \mathbf{A}(t)\mathbf{\Psi}(t)\,.$$

Because $\det(\mathbf{\Psi}(t)) = \mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = 3 \neq 0$, we see that $\mathbf{\Psi}(t)$ is a fundamental matrix of the linear system with $\mathbf{A}(t)$ given by

$$\begin{aligned} \mathbf{A}(t) = \mathbf{\Psi}'(t)\,\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 0 & 3t^2 \\ 1 & 4t^3 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 3+t^4 & -t^3 \\ -t & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} -3t^3 & 3t^2 \\ 3-3t^4 & 3t^3 \end{pmatrix} = \begin{pmatrix} -t^3 & t^2 \\ 1-t^4 & t^3 \end{pmatrix}. \end{aligned}$$

Therefore $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions for the linear system whose coefficient matrix is this $\mathbf{A}(t)$.

Solution (c). Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions for the linear system whose coefficient matrix is $\mathbf{A}(t)$, a general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} 1 \\ t \end{pmatrix} + c_2 \begin{pmatrix} t^3 \\ 3+t^4 \end{pmatrix}.$$

---

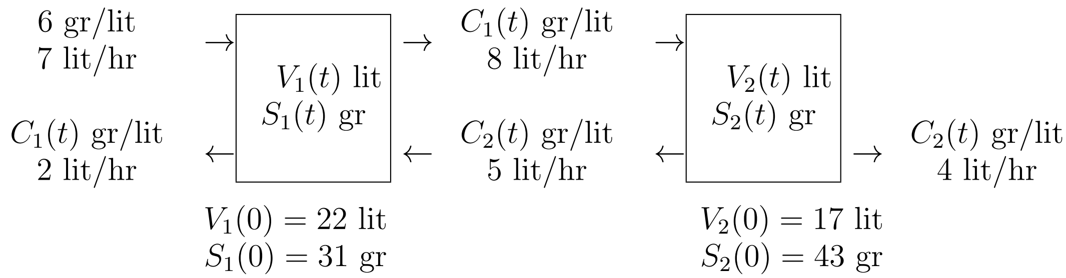
(19) Two interconnected tanks, each with a capacity of 60 liters, contain brine (salt water). At $t = 0$ the first tank contains 22 liters and the second contains 17 liters. Brine with a salt concentration of 6 grams per liter flows into the first tank at 7 liters per hour. Well-stirred brine flows from the first tank into the second at 8 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 2 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 31 grams of salt in the first tank and 43 grams in the second.

(a) Determine the volume of brine in each tank as a function of time.

(b) Give an initial-value problem that governs the amount of salt in each tank as a function of time. (Do not solve the IVP.)

(c) Give the interval of definition for the solution of this initial-value problem.

Remark. Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t hours. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t hours. Because the mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.



Solution (a). You are asked to determine $V_1(t)$ and $V_2(t)$. The rates work out so that

$$V_1(t) = 22 + 2t \quad \text{liters}\,, \qquad V_2(t) = 17 - t \quad \text{liters}\,.$$

Remark. Because the tanks each have a capacity of 60 liters, we have the restrictions

$$0 \le V_1(t) = 22 + 2t \le 60\,, \qquad 0 \le V_2(t) = 17 - t \le 60\,.$$

These restrictions are

$$-11 \le t \le 19\,, \qquad -53 \le t \le 17\,,$$

which combine to give the restrictions

$$-11 \le t \le 17\,.$$

Notice that each of these restrictions happen when one of the tanks is empty. If the numbers had been different restriction then one or both of these restrictions could happen when one of the tanks is full. For example, if the tanks each had a capacity of 50 liters then the restrictions would be

$$-11 \le t \le 14\,.$$

Solution (b). You are asked to give an initial-value problem that governs $S_1(t)$ and $S_2(t)$. These are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 6{\cdot}7 + \frac{S_2}{17-t}\,5 - \frac{S_1}{22+2t}\,8 - \frac{S_1}{22+2t}\,2\,, \qquad & S_1(0) = 31\,,\\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{22+2t}\,8 - \frac{S_2}{17-t}\,5 - \frac{S_2}{17-t}\,4\,, \qquad & S_2(0) = 43\,. \end{aligned}$$

You could leave the answer in the above form. It can however be simplified to

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 42 + \frac{5}{17-t}S_2 - \frac{5}{11+t}S_1\,, \qquad & S_1(0) = 31\,,\\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{4}{11+t}S_1 - \frac{9}{17-t}S_2\,, \qquad & S_2(0) = 43\,. \end{aligned}$$

Solution (c). You are asked to give the interval of definition for the solution of this initial-value problem. This can be done because the differential equation is *linear*. Its coefficients are undefined either at $t = -11$ or at $t = 17$ and are continuous elsewhere. Its forcing is constant, so is continuous everywhere. Therefore the natural interval of definition of this initial-value problem is $(-11, 17)$ because:

- the initial time $t = 0$ is in $(-11, 17)$;
- all the coefficients and the forcing are continuous over $(-11, 17)$;
- two coefficients are undefined at $t = -11$;
- two coefficients are undefined at $t = 17$.

This interval is consistent with the restictions given earlier. However, it could also be argued that the interval of definition for the solution of this initial-value problem is $[0, 17)$ because the word problem starts at $t = 0$.

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(20) Give a real, vector-valued general solution of the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for

$$\text{(a)} \quad \mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}, \qquad\qquad \text{(b)} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Solution (a) by Eigen Methods. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 6z - 16 = (z+2)(z-8). \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -2 and 8. We can see from the nonzero columns of the matrices

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}, \qquad \mathbf{A} - 8\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix},$$

that $\mathbf{A}$ has the eigenpairs

$$\left(-2, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right), \qquad \left(8, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

Form these eigenpairs we construct the solutions

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{8t}\begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

Therefore a general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{8t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solution (a) by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 6z - 16 = (z-3)^2 - 25 = (z-3)^2 - 5^2. \end{aligned}$$

This is a difference of squares with $\mu = 3$ and $\nu = 5$. Therefore we have

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t}\left[\cosh(5t)\mathbf{I} + \frac{\sinh(5t)}{5}(\mathbf{A} - 3\mathbf{I})\right] \\ &= e^{3t}\left[\cosh(5t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\sinh(5t)}{5}\begin{pmatrix}3 & 4\\ 4 & -3\end{pmatrix}\right] \\ &= e^{3t}\begin{pmatrix}\cosh(5t) + \frac{3}{5}\sinh(5t) & \frac{4}{5}\sinh(5t)\\ \frac{4}{5}\sinh(5t) & \cosh(5t) - \frac{3}{5}\sinh(5t)\end{pmatrix}. \end{aligned}$$

Therefore a general solution is given by

$$\mathbf{x}(t) = e^{t\mathbf{A}}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = c_1 e^{3t}\begin{pmatrix}\cosh(5t) + \frac{3}{5}\sinh(5t)\\ \frac{4}{5}\sinh(5t)\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}\frac{4}{5}\sinh(5t)\\ \cosh(5t) - \frac{3}{5}\sinh(5t)\end{pmatrix}.$$

Solution (b) by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2. \end{aligned}$$

This is a sum of squares with $\mu = 1$ and $\nu = 2$. Therefore we have

$$\begin{aligned} e^{t\mathbf{A}} &= e^{t}\left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A} - \mathbf{I})\right] \\ &= e^{t}\left[\cos(2t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\sin(2t)}{2}\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}\right] = e^{t}\begin{pmatrix}\cos(2t) & \sin(2t)\\ -\sin(2t) & \cos(2t)\end{pmatrix}. \end{aligned}$$

Therefore a general solution is given by

$$\mathbf{x}(t) = e^{t\mathbf{A}}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = c_1 e^{t}\begin{pmatrix}\cos(2t)\\ -\sin(2t)\end{pmatrix} + c_2 e^{t}\begin{pmatrix}\sin(2t)\\ \cos(2t)\end{pmatrix}.$$

Solution (b) by Eigen Methods. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. We can see from the nonzero columns of the matrices

$$\mathbf{A} - (1+i2)\mathbf{I} = \begin{pmatrix}-i2 & 2\\ -2 & -i2\end{pmatrix}, \qquad \mathbf{A} - (1-i2)\mathbf{I} = \begin{pmatrix}i2 & 2\\ -2 & i2\end{pmatrix},$$

that $\mathbf{A}$ has the eigenpairs

$$\left(1 + i2, \begin{pmatrix}1\\ i\end{pmatrix}\right), \qquad \left(1 - i2, \begin{pmatrix}-i\\ 1\end{pmatrix}\right).$$

Because

$$e^{(1+i2)t}\begin{pmatrix}1\\ i\end{pmatrix} = e^{t}\begin{pmatrix}\cos(2t) + i\sin(2t)\\ -\sin(2t) + i\cos(2t)\end{pmatrix},$$

two real solutions of the system are

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

Therefore a general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 e^t \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

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(21) Sketch the phase-plane portrait of the linear planar system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for

(a) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$, (b) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

Solution (a). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ 4 & 0 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z - 16 = (z+2)(z-8).$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -2 and 8. Because $\mathbf{A}$ has real eigenvalues of opposite signs, the phase-plane portrait is a *saddle*, which is thereby *unstable*. To sketch the phase-plane portrait we need the eigenpairs of $\mathbf{A}$. We can see from the nonzero columns of the matrices

$$\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}, \qquad \mathbf{A} - 8\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix},$$

that $\mathbf{A}$ has the eigenpairs

$$\left(-2, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right), \qquad \left(8, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right).$$

These show that one orbit moves away from $(0,0)$ along each half of the line $x = \frac{1}{2}y$, and one orbit moves towards $(0,0)$ along each half of the line $y = -2x$. (These are the lines of eigenvectors.) Every other orbit sweeps away from the line $y = -2x$ and towards the line $x = \frac{1}{2}y$. A sketch of the phase-plane portrait will be given during the review session provided someone asks for it.

Solution (b). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ is

$$\begin{aligned} p(z) &= z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) \\ &= z^2 - 2z + 5 = (z-1)^2 + 4 = (z-1)^2 + 2^2. \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. Because $\mathbf{A}$ has a conjugate pair of eigenvalues with positive real part, and because $a_{21} = -2 < 0$, the phase-plane portrait is a *clockwise spiral source*, which is thereby *repelling*. A sketch of the phase-plane portrait will be given during the review session provided someone asks for it.

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(22) What answer will be produced by the following Matlab command?

$$>> \text{A} = [1\ 4;\ 3\ 2];\ [\text{vect, val}] = \text{eig}(\text{sym}(\text{A}))$$

You do not have to give the answer in Matlab format.

Solution. The Matlab command will produce the eigenpairs of $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \text{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 3z - 10 = (z-5)(z+2)\,,$$

so its eigenvalues are 5 and -2. We can see from the nonzero columns of the matrices

$$\mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}, \qquad \mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix},$$

that $\mathbf{A}$ has eigenpairs

$$\left(5\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \qquad \left(-2\,, \begin{pmatrix} -4 \\ 3 \end{pmatrix}\right).$$

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(23) A real 2×2 matrix $\mathbf{B}$ has the eigenpairs

$$\left(2\,, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right) \quad \text{and} \quad \left(-1\,, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right).$$

(a) Give a general solution to the linear planar system $\mathbf{x}' = \mathbf{B}\mathbf{x}$.
(b) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ that diagonalize $\mathbf{B}$.
(c) Compute $e^{t\mathbf{B}}$.
(d) Sketch a phase-plane portrait for this system and identify its type. Classify the stability of the origin. Carefully mark all sketched orbits with arrows!

Solution (a). Use the given eigenpairs to construct the solutions

$$\mathbf{x}_1(t) = e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Therefore a general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Solution (b). The matrix $\mathbf{B}$ can be diagonalized by

$$\mathbf{V} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solution (c). By part (b) we have $\mathbf{B} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ where

$$\mathbf{V} = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{V}^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}.$$

Then

$$\begin{aligned} e^{t\mathbf{B}} &= \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} = \frac{1}{7}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} 3e^{2t} & -e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 6e^{2t} + e^{-t} & 3e^{2t} - 3e^{-t} \\ 2e^{2t} - 2e^{-t} & e^{2t} + 6e^{-t} \end{pmatrix}. \end{aligned}$$

Alternative Solution (c). By part (a) a fundamental matrix is

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} 3e^{2t} & -e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}.$$

Then

$$\begin{aligned} e^{t\mathbf{B}} &= \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(0)^{-1} = \begin{pmatrix} 3e^{2t} & -e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}^{-1} \\ &= \frac{1}{7}\begin{pmatrix} 3e^{2t} & -e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 6e^{2t} + e^{-t} & 3e^{2t} - 3e^{-t} \\ 2e^{2t} - 2e^{-t} & e^{2t} + 6e^{-t} \end{pmatrix}. \end{aligned}$$

Solution (d). The matrix $\mathbf{B}$ has two real eigenvalues of opposite sign. Therefore the origin is a *saddle* and is thereby *unstable*. There is one orbit moves away from $(0, 0)$ along each half of the line $x = 3y$, and one orbit moves towards $(0, 0)$ along each half of the line $y = -2x$. (These are the lines of eigenvectors.) Every other orbit sweeps away from the line $y = -2x$ and towards the line $x = 3y$. A phase-plane portrait might be sketched during the review session.

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(24) Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\quad \mathbf{x}(0) = \mathbf{x}^{\mathrm{I}}$ for the following $\mathbf{A}$ and $\mathbf{x}^{\mathrm{I}}$.

(a) $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ -5 & -7 \end{pmatrix}, \qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$

(b) $\mathbf{A} = \begin{pmatrix} 8 & -5 \\ 5 & -2 \end{pmatrix}, \qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$

(c) $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}, \qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$

Solution (a) by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ -5 & -7 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 29 = (z+2)^2 + 5^2.$$

This is a sum of squares with $\mu = -2$ and $\nu = 5$. Therefore we have

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-2t}\left[\cos(5t)\mathbf{I} + \frac{\sin(5t)}{5}(\mathbf{A} + 2\mathbf{I})\right] \\ &= e^{-2t}\left[\cos(5t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(5t)}{5}\begin{pmatrix} 5 & 10 \\ -5 & -5 \end{pmatrix}\right] \\ &= e^{-2t}\begin{pmatrix} \cos(5t) + \sin(5t) & 2\sin(5t) \\ -\sin(5t) & \cos(5t) - \sin(5t) \end{pmatrix}. \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned}\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^{\mathrm{I}} &= e^{-2t}\begin{pmatrix}\cos(5t)+\sin(5t) & 2\sin(5t)\\ -\sin(5t) & \cos(5t)-\sin(5t)\end{pmatrix}\begin{pmatrix}-3\\ 2\end{pmatrix}\\ &= e^{-2t}\begin{pmatrix}-3\cos(5t)+\sin(5t)\\ 2\cos(5t)+\sin(5t)\end{pmatrix}.\end{aligned}$$

Solution (a) by Eigen Methods. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}3 & 10\\ -5 & -7\end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 29 = (z+2)^2 + 5^2\,.$$

Therefore the eigenvlues of $\mathbf{A}$ are $-2 \pm i5$. We can see from the nonzero columns of the matrix

$$\mathbf{A} - (-2+i5)\mathbf{I} = \begin{pmatrix}5-i5 & 10\\ -5 & -5-i5\end{pmatrix},$$

that $\mathbf{A}$ has the conjugate eigenpairs

$$\left(-2+i5\,,\ \begin{pmatrix}1+i\\ -1\end{pmatrix}\right), \qquad \left(-2-i5\,,\ \begin{pmatrix}1-i\\ -1\end{pmatrix}\right).$$

Because

$$\begin{aligned}e^{-2t+i5t}\begin{pmatrix}1+i\\ -1\end{pmatrix} &= e^{-2t}\big(\cos(5t)+i\sin(5t)\big)\begin{pmatrix}1+i\\ -1\end{pmatrix}\\ &= e^{-2t}\begin{pmatrix}\cos(5t)-\sin(5t)+i\cos(5t)+i\sin(5t)\\ -\cos(5t)-i\sin(5t)\end{pmatrix},\end{aligned}$$

a fundamental set of real solutions is

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix}\cos(5t)-\sin(5t)\\ -\cos(5t)\end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{-2t}\begin{pmatrix}\cos(5t)+\sin(5t)\\ -\sin(5t)\end{pmatrix}.$$

Then a fundamental matrix $\mathbf{\Psi}(t)$ is given by

$$\mathbf{\Psi}(t) = \big(\mathbf{x}_1(t) \quad \mathbf{x}_2(t)\big) = e^{-2t}\begin{pmatrix}\cos(5t)-\sin(5t) & \cos(5t)+\sin(5t)\\ -\cos(5t) & -\sin(5t)\end{pmatrix}.$$

Because

$$\mathbf{\Psi}(0)^{-1} = \begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix},$$

we see that

$$\begin{aligned}e^{t\mathbf{A}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} &= e^{-2t}\begin{pmatrix}\cos(5t)-\sin(5t) & \cos(5t)+\sin(5t)\\ -\cos(5t) & -\sin(5t)\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}\\ &= e^{-2t}\begin{pmatrix}\cos(5t)+\sin(5t) & 2\sin(5t)\\ -\sin(5t) & \cos(5t)-\sin(5t)\end{pmatrix}.\end{aligned}$$

Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^{\mathrm{I}} = e^{-2t}\begin{pmatrix}\cos(5t)+\sin(5t) & 2\sin(5t)\\ -\sin(5t) & \cos(5t)-\sin(5t)\end{pmatrix}\begin{pmatrix}-3\\ 2\end{pmatrix}$$
$$= e^{-2t}\begin{pmatrix}-3\cos(5t)+\sin(5t)\\ 2\cos(5t)+\sin(5t)\end{pmatrix}.$$

Remark. After we have constructed the fundmental set of solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, we could also have solved the initial-value problem by finding constants c_1 and c_2 such that $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ satisfies the initial condition. Had we done this using the $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ constructed above, we would find that $c_1 = -2$ and $c_2 = -1$.

Solution (b) by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}8 & -5\\ 5 & -2\end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 9 = (z-3)^2.$$

This is a perfect square with $\mu = 3$. Therefore we have

$$e^{t\mathbf{A}} = e^{3t}\big[\mathbf{I} + t\,(\mathbf{A} - 3\mathbf{I})\big] = e^{3t}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + t\begin{pmatrix}5 & -5\\ 5 & -5\end{pmatrix}\right] = e^{3t}\begin{pmatrix}1+5t & -5t\\ 5t & 1-5t\end{pmatrix}.$$

Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^{\mathrm{I}} = e^{3t}\begin{pmatrix}1+5t & -5t\\ 5t & 1-5t\end{pmatrix}\begin{pmatrix}3\\ -1\end{pmatrix} = e^{3t}\begin{pmatrix}3+20t\\ -1+20t\end{pmatrix}.$$

This solution grows like $20t\,e^{3t}$ as $t \to \infty$.

Solution (b) by Eigen Methods. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}8 & -5\\ 5 & -2\end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 9 = (z-3)^2.$$

The only eigenvalue of $\mathbf{A}$ is 3. We can see from the nonzero columns of the matrix

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix}5 & -5\\ 5 & -5\end{pmatrix},$$

that $\mathbf{A}$ has the eigenpair

$$\left(3,\ \begin{pmatrix}1\\ 1\end{pmatrix}\right).$$

We can use this eigenpair to construct the solution

$$\mathbf{x}_1(t) = e^{3t}\begin{pmatrix}1\\ 1\end{pmatrix}.$$

A second solution can be constructed by

$$\mathbf{x}_2(t) = e^{3t}\mathbf{w} + t\,e^{3t}(\mathbf{A} - 3\mathbf{I})\mathbf{w},$$

where $\mathbf{w}$ is any nonzero vector that is not an eigenvector associated with 3. For example, taking $\mathbf{w} = \begin{pmatrix}1 & 0\end{pmatrix}^T$ yields

$$\mathbf{x}_2(t) = e^{3t}\begin{pmatrix}1\\ 0\end{pmatrix} + t\,e^{3t}\begin{pmatrix}5 & -5\\ 5 & -5\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = e^{3t}\begin{pmatrix}1+5t\\ 5t\end{pmatrix}.$$

Then a fundamental matrix $\mathbf{\Psi}(t)$ is given by

$$\mathbf{\Psi}(t) = \big(\mathbf{x}_1(t) \quad \mathbf{x}_2(t)\big) = e^{3t}\begin{pmatrix} 1 & 1+5t \\ 1 & 5t \end{pmatrix}.$$

Because

$$\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

we see that

$$e^{t\mathbf{A}} = \mathbf{\Psi}(t)\mathbf{\Psi}(0)^{-1} = \begin{pmatrix} 1 & 1+5t \\ 1 & 5t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+5t & -5t \\ 5t & 1-5t \end{pmatrix}.$$

Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^{\mathrm{I}} = e^{3t}\begin{pmatrix} 1+5t & -5t \\ 5t & 1-5t \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = e^{3t}\begin{pmatrix} 3+20t \\ -1+20t \end{pmatrix}.$$

This solution grows like $20t\, e^{3t}$ as $t \to \infty$.

Remark. After we have constructed the fundmental set of solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, we could also have solved the initial-value problem by finding constants c_1 and c_2 such that $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ satisfies the initial condition. Had we done this using the $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ constructed above, we would find that $c_1 = -1$ and $c_2 = 4$.

Solution (c) by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 6z + 9 = (z+3)^2.$$

This is a perfect square with $\mu = -3$. Therefore we have

$$e^{t\mathbf{A}} = e^{-3t}\big[\mathbf{I} + t\,(\mathbf{A} + 3\mathbf{I})\big] = e^{-3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right] = e^{-3t}\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}.$$

Therefore the solution of the initial-value problem is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^{\mathrm{I}} = e^{-3t}\begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = e^{-3t}\begin{pmatrix} 3+4t \\ 1-4t \end{pmatrix}.$$

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(25) Consider the system

$$\dot{x} = 2xy, \qquad \dot{y} = 9 - 9x - y^2.$$

(a) Find all of its stationary points.
(b) Find all of its semistationary orbits.
(c) Find a nonconstant function $H(x, y)$ such that every orbit of the system satisfies $H(x, y) = c$ for some constant c.
(d) Classify the type and stability of each stationary point.
(e) Sketch the stationary points plus the level set $H(x, y) = c$ for each value of c that corresponds to a stationary point that is a saddle. Carefully mark all sketched orbits with arrows!

Solution (a). Stationary points satisfy

$$0 = 2xy\,,$$
$$0 = 9 - 9x - y^2\,.$$

The top equation shows that $x = 0$ or $y = 0$. If $x = 0$ then the bottom equation becomes $0 = 9 - y^2 = (3 - y)(3 + y)$, which shows that either $y = 3$ or $y = -3$. If $y = 0$ then the bottom equation becomes $0 = 9 - 9x = 9(1 - x)$, which shows that $x = 1$. Therefore the stationary points of the system are

$$(0,3)\,, \qquad (0,-3)\,, \qquad (1,0)\,.$$

Solution (b). When $x = 0$ the right-hand side of the $\dot{x}$ equation is zero for every y. So the system has semistationary solutions of the form $(0, Y(t))$ when $Y(t)$ is any nonstationary solution of

$$\dot{y} = 9 - y^2\,.$$

The orbits of these solutions all lie on the line $x = 0$, which is the y-axis.

There is no value of y that makes the right-hand side of the $\dot{y}$ equation is zero for every x. So the system has no semistationary solutions of the form $(X(t), b)$.

Therefore the semistationary orbits are the three open line segments of the y-axis that are separated by the stationary points $(0, -3)$ and $(0, 3)$.

Solution (c). The associated first-order orbit equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9 - 9x - y^2}{2xy}\,.$$

This equation is not linear or separable. It has the differential form

$$(y^2 + 9x - 9)\,\mathrm{d}x + 2xy\,\mathrm{d}y = 0\,,$$

which is exact because

$$\partial_y(y^2 + 9x - 9) = 2y \qquad = \qquad \partial_x(2xy) = 2y\,.$$

Therefore there exists $H(x, y)$ such that

$$\partial_x H(x,y) = y^2 + 9x - 9\,, \qquad \partial_y H(x,y) = 2xy\,.$$

By integrating the second equation we see that

$$H(x,y) = xy^2 + h(x)\,.$$

When this is substituted into the first equation we find

$$\partial_x H(x,y) = y^2 + h'(x) = y^2 + 9x - 9\,,$$

which implies that $h'(x) = 9x - 9$. By taking $h(x) = \frac{9}{2}x^2 - 9x$ we obtain

$$H(x,y) = xy^2 + \tfrac{9}{2}x^2 - 9x\,.$$

Alternative Solution (c). Notice that

$$\partial_x f(x,y) + \partial_y g(x,y) = \partial_x(2xy) + \partial_y(9 - 9x - y^2) = 2y - 2y = 0\,.$$

Therefore the system has Hamiltonian form with Hamiltonian $H(x, y)$ that satisfies

$$\partial_y H(x,y) = 2xy\,, \qquad -\partial_x H(x,y) = 9 - 9x - y^2\,.$$

By integrating the first equation we see that

$$H(x,y) = xy^2 + h(x)\,.$$

When this is substituted into the second equation we find

$$-\partial_x H(x,y) = -y^2 - h'(x) = 9 - 9x - y^2\,,$$

which implies that $h'(x) = 9x - 9$. By taking $h(x) = \frac{9}{2}x^2 - 9x$ we obtain

$$H(x,y) = xy^2 + \tfrac{9}{2}x^2 - 9x\,.$$

Solution (d). Because

$$\mathbf{f}(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 2xy \\ 9 - 9x - y^2 \end{pmatrix}\,,$$

the Jacobian matrix $\boldsymbol{\partial}\mathbf{f}(x,y)$ of partial derivatives is

$$\boldsymbol{\partial}\mathbf{f}(x,y) = \begin{pmatrix} \partial_x f(x,y) & \partial_y f(x,y) \\ \partial_x g(x,y) & \partial_y g(x,y) \end{pmatrix} = \begin{pmatrix} 2y & 2x \\ -9 & -2y \end{pmatrix}\,.$$

Evaluating this matrix at each stationary point yields

$$\boldsymbol{\partial}\mathbf{f}(0,3) = \begin{pmatrix} 6 & 0 \\ -9 & -6 \end{pmatrix}\,, \qquad \boldsymbol{\partial}\mathbf{f}(0,-3) = \begin{pmatrix} -6 & 0 \\ -9 & 6 \end{pmatrix}\,, \qquad \boldsymbol{\partial}\mathbf{f}(1,0) = \begin{pmatrix} 0 & 2 \\ -9 & 0 \end{pmatrix}\,.$$

- Because the matrix $\boldsymbol{\partial}\mathbf{f}(0,3)$ is lower triangular, we can read off that its eigenvalues are 6 and -6. Because these are real with opposite signs, the stationary point $(0,3)$ is a *saddle* and thereby is *unstable*.
- Because the matrix $\boldsymbol{\partial}\mathbf{f}(0,-3)$ is lower triangular, we can read off that its eigenvalues are -6 and 6. Because these are real with opposite signs, the stationary point $(0,-3)$ is a *saddle* and thereby is *unstable*.
- The characteristic polynomial of the matrix $\boldsymbol{\partial}\mathbf{f}(1,0)$ is

  $$p(z) = z^2 + 18\,,$$

  so the matrix $\boldsymbol{\partial}\mathbf{f}(1,0)$ has eigenvalues $\pm i\sqrt{18}$. Because these are immaginary and the system has an integral while the lower left entry of $\boldsymbol{\partial}\mathbf{f}(1,0)$ is negative, the stationary point $(1,0)$ is a *clockwise center* and thereby is *stable*.

Alternative Solution (b). If you saw that the system has Hamiltonian form with Hamiltonian $H(x,y)$ from part (b) then you can take this approach. The Hessian matrix $\boldsymbol{\partial}^2 H(x,y)$ of second partial derivatives of the Hamiltonian $H(x,y)$ is

$$\boldsymbol{\partial}^2 H(x,y) = \begin{pmatrix} \partial_{xx} H(x,y) & \partial_{xy} H(x,y) \\ \partial_{yx} H(x,y) & \partial_{yy} H(x,y) \end{pmatrix} = \begin{pmatrix} 9 & 2y \\ 2y & 2x \end{pmatrix}\,.$$

Evaluating this at the stationary points yields

$$\boldsymbol{\partial}^2 H(0,3) = \begin{pmatrix} 9 & 6 \\ 6 & 0 \end{pmatrix}\,, \qquad \boldsymbol{\partial}^2 H(0,-3) = \begin{pmatrix} 9 & -6 \\ -6 & 0 \end{pmatrix}\,, \qquad \boldsymbol{\partial}^2 H(1,0) = \begin{pmatrix} 9 & 0 \\ 0 & 2 \end{pmatrix}\,.$$

- The characteristic polynomial of the matrix $\boldsymbol{\partial}^2 H(0,3)$ is
$$p(z) = z^2 - 9z - 36 = (z-12)(z+3)\,.$$
Therefore the matrix $\boldsymbol{\partial}^2 H(0,3)$ has eigenvalues 12 and -3. Because these have different signs, the stationary point $(0,3)$ is a *saddle* and thereby is *unstable*.
- The characteristic polynomial of the matrix $\boldsymbol{\partial}^2 H(0,-3)$ is
$$p(z) = z^2 - 9z - 36 = (z-12)(z+3)\,.$$
Therefore the matrix $\boldsymbol{\partial}^2 H(0,-3)$ has eigenvalues 12 and -3. Because these have different signs, the stationary point $(0,-3)$ is a *saddle* and thereby is *unstable*.
- Because the matrix $\boldsymbol{\partial}^2 H(1,0)$ is diagonal, we can read off that its eigenvalues are 9 and 2. Because these are both positive, the stationary point $(1,0)$ is a *clockwise center* and thereby is *stable*.

Solution (e). The saddle points are $(0,3)$ and $(0,-3)$. Because
$$H(0,3) = H(0,-3) = 0 \cdot (\pm 3)^2 + \tfrac{9}{2} \cdot 0^2 - 9 \cdot 0 = 0\,.$$
Hence, the level set corresponding to these saddle points is
$$0 = xy^2 + \tfrac{9}{2}x^2 - 9x = (y^2 + \tfrac{9}{2}x - 9)x\,.$$
The points on this set must satisfy either $y^2 + \frac{9}{2}x - 9 = 0$ or $x = 0$. Therefore the level set is the union of the parabola $x = 2 - \frac{2}{9}y^2$ and the y-axis.

Along the y-axis ($x = 0$) the $\dot{y}$ equation reduces to $\dot{y} = 9 - y^2 = (3-y)(3+y)$, whereby the arrows point towards $(0,3)$ and away from $(0,-3)$. Along the parabola $x = 2 - \frac{2}{9}y^2$ the arrows point away from $(0,3)$ and towards $(0,-3)$ because they are saddle points.

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(26) Consider the system
$$\dot{p} = -9p + 3q\,, \qquad \dot{q} = 4p - 8q + 10p^2\,.$$
(a) This system has two stationary points. Find them.
(b) Find the Jacobian matrix at each stationary point.
(c) Classify the type and stability of each stationary point.
(d) Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. Carefully mark all sketched orbits with arrows!

Solution (a). Stationary points satisfy
$$0 = -9p + 3q\,, \qquad 0 = 4p - 8q + 10p^2\,.$$
The first equation implies $q = 3p$, whereby the second equation becomes $0 = -20p + 10p^2 = 10p(p-2)$, which implies either $p = 0$ or $p = 2$. Therefore all the stationary points of the system are
$$(0,0)\,, \qquad (2,6)\,.$$

Solution (b). Because
$$\mathbf{f}(p,q) = \begin{pmatrix} f(p,q) \\ g(p,q) \end{pmatrix} = \begin{pmatrix} -9p + 3q \\ 4p - 8q + 10p^2 \end{pmatrix}\,,$$

the Jacobian matrix of partial derivatives is

$$\boldsymbol{\partial}\mathbf{f}(u,v) = \begin{pmatrix} \partial_p f(p,q) & \partial_q f(p,q) \\ \partial_p g(p,q) & \partial_q g(p,q) \end{pmatrix} = \begin{pmatrix} -9 & 3 \\ 4+20p & -8 \end{pmatrix} .$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} -9 & 3 \\ 4 & -8 \end{pmatrix} \quad \text{at } (0,0) , \qquad \mathbf{B} = \begin{pmatrix} -9 & 3 \\ 44 & -8 \end{pmatrix} \quad \text{at } (2,6) .$$

Solution (c). The coefficient matrix $\mathbf{A}$ at $(0,0)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{B}) = z^2 - \operatorname{tr}(\mathbf{B})z + \det(\mathbf{B}) = z^2 + 17z + 60 = (z+5)(z+12) .$$

The eigenvalues are -5 and -12, whereby the stationary point $(0,0)$ is a *nodal sink*, which is *attracting*. This describes the phase-plane portrait of the nonlinear system near $(0,0)$.

The coefficient matrix $\mathbf{B}$ at $(2,6)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 17z - 60 = (z-3)(z+20) .$$

The eigenvalues are -20 and 3, whereby the stationary point $(0,0)$ is a *saddle*, which is *unstable*. This describes the phase-plane portrait of the nonlinear system near $(2,6)$.

Solution (d). The stationary point $(0,0)$ is a *nodal sink*. The coefficient matrix $\mathbf{A}$ has eigenvalues -5 and -12. We can see from the nonzero columns of the matrices

$$\mathbf{A} + 5\mathbf{I} = \begin{pmatrix} -4 & 3 \\ 4 & -3 \end{pmatrix} , \qquad \mathbf{A} + 12\mathbf{I} = \begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix} ,$$

that $\mathbf{A}$ has the eigenpairs

$$\left(-5 \, , \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right) , \qquad \left(-12 \, , \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$

Near $(2,6)$ there is one orbit that approaches $(0,0)$ tangent to each side of the line $q = -p$. Every other orbit near $(0,0)$ approaches $(0,0)$ tangent to one side of the line $q = \frac{4}{3}p$.

The stationary point $(2,6)$ is a *saddle*. The coefficient matrix $\mathbf{B}$ has eigenvalues -20 and 3. We can see from the nonzero columns of the matrices

$$\mathbf{B} + 20\mathbf{I} = \begin{pmatrix} 11 & 3 \\ 44 & 12 \end{pmatrix} , \qquad \mathbf{B} - 3\mathbf{I} = \begin{pmatrix} -12 & 3 \\ 44 & -11 \end{pmatrix} ,$$

that $\mathbf{B}$ has the eigenpairs

$$\left(-20 \, , \begin{pmatrix} 3 \\ -11 \end{pmatrix} \right) , \qquad \left(3 \, , \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right)$$

Near $(2,6)$ there is one orbit that emerges from $(2,6)$ tangent to each side of the line $q - 6 = 4(p-2)$. There is also one orbit that approaches $(2,6)$ tangent to each side of the line $q - 6 = -\frac{11}{3}(p-2)$. These orbits are separatrices. A phase-plane portrait might be sketched during the review session.

(27) Consider the system

$$u' = -5v\,, \qquad v' = u - 4v - u^2\,.$$

(a) Find all of its stationary points.
(b) Compute the Jacobian matrix at each stationary point.
(c) Classify the type and stability of each stationary point.
(d) Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. Carefully mark all sketched orbits with arrows!

Solution (a). Stationary points satisfy

$$0 = -5v\,, \qquad 0 = u - 4v - u^2\,.$$

The first equation implies $v = 0$, whereby the second equation becomes $0 = u - u^2 = u(1-u)$, which implies either $u = 0$ or $u = 1$. Therefore all the stationary points of the system are

$$(0,0)\,, \qquad (1,0)\,.$$

Solution (b). Because

$$\mathbf{f}(u,v) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = \begin{pmatrix} -5v \\ u - 4v - u^2 \end{pmatrix}\,,$$

the Jacobian matrix of partial derivatives is

$$\boldsymbol{\partial}\mathbf{f}(u,v) = \begin{pmatrix} \partial_u f(u,v) & \partial_v f(u,v) \\ \partial_u g(u,v) & \partial_v g(u,v) \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 1 - 2u & -4 \end{pmatrix}\,.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} 0 & -5 \\ 1 & -4 \end{pmatrix} \quad \text{at } (0,0)\,, \qquad \mathbf{B} = \begin{pmatrix} 0 & -5 \\ -1 & -4 \end{pmatrix} \quad \text{at } (1,0)\,.$$

Solution (c). The coefficient matrix $\mathbf{A}$ at $(0,0)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 5 = (z+2)^2 + 1^2\,.$$

The eigenvalues are thereby $-2 \pm i$. Because $a_{21} = 1 > 0$, the stationary point $(0,0)$ is a *counterclockwise spiral sink*, which is *attracting*. This describes the phase-plane portrait of the nonlinear system near $(0,0)$.

The coefficient matrix $\mathbf{B}$ at $(1,0)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{B}) = z^2 - \operatorname{tr}(\mathbf{B})z + \det(\mathbf{B}) = z^2 + 4z - 5 = (z+2)^2 - 3^2\,.$$

The eigenvalues are thereby -2 ± 3, or simply -5 and 1. The stationary point $(1,0)$ is thereby a *saddle*, which is *unstable*. This describes the phase-plane portrait of the nonlinear system near $(1,0)$.

Solution (d). The stationary point $(0,0)$ is a *counterclockwise spiral sink*.

The stationary point $(1,0)$ is a *saddle*. The coefficient matrix $\mathbf{B}$ has eigenvalues -5 and 1. We can see from the nonzero columns of the matrices

$$\mathbf{B} + 5\mathbf{I} = \begin{pmatrix} 5 & -5 \\ -1 & 1 \end{pmatrix}\,, \qquad \mathbf{B} - \mathbf{I} = \begin{pmatrix} -1 & -5 \\ -1 & -5 \end{pmatrix}\,,$$

that $\mathbf{B}$ has the eigenpairs

$$\left(-5\,,\begin{pmatrix}1\\1\end{pmatrix}\right)\,,\qquad \left(1\,,\begin{pmatrix}-5\\1\end{pmatrix}\right)$$

Near $(1,0)$ there is one orbit that emerges from $(1,0)$ tangent to each side of the line $u = 1 - 5v$. There is also one orbit that approaches $(1,0)$ tangent to each side of the line $v = u - 1$. These orbits are separatrices. A phase-plane portrait might be sketched during the review session.

Remark. The global phase-plane portrait becomes clearer if we had seen that $H(u,v) = \frac{1}{2}u^2 + \frac{5}{2}v^2 - \frac{1}{3}u^3$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(u,v) &= \partial_u H(u,v)\,u' + \partial_v H(u,v)\,v' \\ &= (u-u^2)(-5v) + 5v(u-4v-u^2) = -20v^2 \le 0\,.\end{aligned}$$

The orbits of the system are thereby seen to cross the level sets of $H(u,v)$ so as to decrease $H(x,y)$. You would *not* be expected to see this on the Final Exam.

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(28) Consider the system

$$\dot{p} = p(3 - 3p + 2q)\,,\qquad \dot{q} = q(6 - p - q)\,.$$

(a) Find all of its stationary points.
(b) Compute the Jacobian matrix at each stationary point.
(c) Classify the type and stability of each stationary point.
(d) Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. Carefully mark all sketched orbits with arrows!
(e) Add the orbits of all semistationary solutions to the phase-plane portrait sketched for part (d). Carefully mark these sketched orbits with arrows!
(f) Why do solutions that start in the first quadrant stay in the first quadrant?

Solution (a). Stationary points satisfy

$$0 = p(3 - 3p + 2q)\,,\qquad 0 = q(6 - p - q)\,.$$

The first equation implies either $p = 0$ or $3 - 3p + 2q = 0$, while the second equation implies either $q = 0$ or $6 - p - q = 0$.

- If $p = 0$ and $q = 0$ then $(0,0)$ is a stationary point.
- If $p = 0$ and $6 - p - q = 0$ then $(0,6)$ is a stationary point.
- If $3 - 3p + 2q = 0$ and $q = 0$ then $(1,0)$ is a stationary point.
- If $3 - 3p + 2q = 0$ and $6 - p - q = 0$ then upon solving these equations we find that $(3,3)$ is a stationary point.

Therefore all the stationary points of the system are

$$(0,0)\,,\qquad (0,6)\,,\qquad (1,0)\,,\qquad (3,3)\,.$$

Solution (b). Because

$$\mathbf{f}(p,q) = \begin{pmatrix} f(p,q) \\ g(p,q) \end{pmatrix} = \begin{pmatrix} 3p - 3p^2 + 2pq \\ 6q - pq - q^2 \end{pmatrix}\,,$$

the Jacobian matrix of partial derivatives is

$$\boldsymbol{\partial}\mathbf{f}(p,q) = \begin{pmatrix} \partial_p f(p,q) & \partial_q f(p,q) \\ \partial_p g(p,q) & \partial_q g(p,q) \end{pmatrix} = \begin{pmatrix} 3-6p+2q & 2p \\ -q & 6-p-2q \end{pmatrix}.$$

Evaluating this matrix at each stationary point yields the coefficient matrices

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \quad \text{at } (0,0)\,, \qquad \mathbf{A} = \begin{pmatrix} 15 & 0 \\ -6 & -6 \end{pmatrix} \quad \text{at } (0,6)\,,$$
$$\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 0 & 5 \end{pmatrix} \quad \text{at } (1,0)\,, \qquad \mathbf{A} = \begin{pmatrix} -9 & 6 \\ -3 & -3 \end{pmatrix} \quad \text{at } (3,3)\,.$$

Solution (c). The coefficient matrix $\mathbf{A}$ at $(0,0)$ is diagonal, so we can read-off its eigenvalues as 3 and 6. The stationary point $(0,0)$ is thereby a *nodal source*, which is *repelling*. This describes the phase-plane portrait of the nonlinear system near $(0,0)$.

The coefficient matrix $\mathbf{A}$ at $(0,6)$ is triangular, so we can read-off its eigenvalues as -6 and 15. The stationary point $(0,6)$ is thereby a *saddle*, which is *unstable*. This describes the phase-plane portrait of the nonlinear system near $(0,6)$.

The coefficient matrix $\mathbf{A}$ at $(1,0)$ is triangular, so we can read-off its eigenvalues as -3 and 5. The stationary point $(1,0)$ is thereby a *saddle*, which is *unstable*. This describes the phase-plane portrait of the nonlinear system near $(1,0)$.

The coefficient matrix $\mathbf{A}$ at $(3,3)$ has eigenvalues that satisfy

$$0 = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 12z + 45 = (z+6)^2 + 3^2\,.$$

Its eigenvalues are thereby $-6 \pm i3$. Because $a_{21} = -3 < 0$, the stationary point $(3,3)$ is a *clockwise spiral sink*, which is *attracting*. This describes the phase-plane portrait of the nonlinear system near $(3,3)$.

Solution (d). The stationary point $(0,0)$ is a *nodal source*. The coefficient matrix $\mathbf{A}$ has eigenvalues 3 and 6. We can see from the nonzero columns of the matrices

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \mathbf{A} - 6\mathbf{I} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix},$$

that $\mathbf{A}$ has the eigenpairs

$$\left(3\,, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right), \qquad \left(6\,, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right).$$

Near $(0,0)$ there is one orbit that emerges from $(0,0)$ along each side of the p-axis and the q-axis. Every other orbit emerges from $(0,0)$ tangent to the p-axis, which is the line corresponding to the eigenvalue with the smaller absolute value.

The stationary point $(0,6)$ is a *saddle*. The coefficient matrix $\mathbf{A}$ has eigenvalues -6 and 15. We can see from the nonzero columns of the matrices

$$\mathbf{A} + 6\mathbf{I} = \begin{pmatrix} 21 & 0 \\ -6 & 0 \end{pmatrix}, \qquad \mathbf{A} - 15\mathbf{I} = \begin{pmatrix} 0 & 0 \\ -6 & -21 \end{pmatrix},$$

that $\mathbf{A}$ has the eigenpairs

$$\left(-6\,, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right), \qquad \left(15\,, \begin{pmatrix} 7 \\ -2 \end{pmatrix}\right)$$

Near $(0,6)$ there is one orbit that approaches $(0,6)$ along each side of the q-axis. There is also one orbit that emerges from $(0,6)$ tangent to each side of the line $q = 6 - \frac{2}{7}p$. These orbits are separatrices.

The stationary point $(1,0)$ is a *saddle*. The coefficient matrix $\mathbf{A}$ has eigenvalues -3 and 5. We can see from the nonzero columns of the matrices

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 0 & 2 \\ 0 & 8 \end{pmatrix}, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -8 & 2 \\ 0 & 0 \end{pmatrix},$$

that $\mathbf{A}$ has the eigenpairs

$$\left(-3, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), \qquad \left(5, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right)$$

Near $(1,0)$ there is one orbit that emerges from $(1,0)$ along each side of the p-axis. There is also one orbit that approaches $(1,0)$ tangent to each side of the line $q = 4(p-1)$. These orbits are also separatrices.

Finally, the stationary point $(3,3)$ is a *clockwise spiral sink*. All orbits in the positive quadrant will spiral into it. A phase-plane portrait might be sketched during the review session.

Solution (e). The lines $p = 0$ and $q = 0$ correspond to semistationary solutions. Along the line $p = 0$ (the q-axis) the system reduces to

$$\dot{q} = q(6 - q).$$

Along the line $q = 0$ (the p-axis) the system reduces to

$$\dot{p} = 3p(1 - p).$$

The arrows along these lines can be determined from a phase-line portrait of these reduced systems. A phase-plane portrait might be sketched during the review session.

Solution (f). Because the p-axis and the q-axis are comprised of semistationary orbits, the uniqueness theorem implies that no other orbits can cross them.

Table of Laplace Transforms

	$h(t) = \mathcal{L}^{-1}[H](t)$		$H(s) = \mathcal{L}[h](s)$	
1.	$t^n e^{at}$	for $n \geq 0$	$\dfrac{n!}{(s-a)^{n+1}}$	for $s > a$
2.	$e^{at}\cos(bt)$		$\dfrac{s-a}{(s-a)^2+b^2}$	for $s > a$
3.	$e^{at}\sin(bt)$		$\dfrac{b}{(s-a)^2+b^2}$	for $s > a$
4.	$e^{at}\cosh(bt)$		$\dfrac{s-a}{(s-a)^2-b^2}$	for $s > a + \lvert b \rvert$
5.	$e^{at}\sinh(bt)$		$\dfrac{b}{(s-a)^2-b^2}$	for $s > a + \lvert b \rvert$
6.	$t^n j(t)$	for $n \geq 0$	$(-1)^n J^{(n)}(s)$	where $J(s) = \mathcal{L}[j](s)$
7.	$j'(t)$		$s\,J(s) - j(0)$	where $J(s) = \mathcal{L}[j](s)$
8.	$e^{at} j(t)$		$J(s-a)$	where $J(s) = \mathcal{L}[j](s)$
9.	$u(t-c)j(t-c)$	for $c \geq 0$	$e^{-cs}J(s)$	where $J(s) = \mathcal{L}[j](s)$
10.	$\delta(t-c)j(t)$	for $c \geq 0$	$e^{-cs}j(c)$	

Here a, b, and c are real numbers; n is an integer; $j(t)$ is any function that is nice enough; $u(t)$ is the unit step (Heaviside) function; $\delta(t)$ is the unit impulse (Dirac delta).