Solutions of the Sample Problems for the Third In-Class Exam
Math 246, Fall 2019, Professor David Levermore

(1) Compute the Laplace transform of $f(t) = t\,e^{3t}u(t-2)$ from its definition.

Solution. The definition of the Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} t\,e^{3t} u(t-2)\,\mathrm{d}t = \lim_{T\to\infty} \int_2^T t\,e^{-(s-3)t}\,\mathrm{d}t\,.$$

This limit diverges to $+\infty$ for $s \leq 3$ because in that case for every $T > 2$ we have

$$\int_2^T t\,e^{-(s-3)t}\,\mathrm{d}t \geq \int_2^T t\,\mathrm{d}t = \frac{T^2}{2} - 2\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

For $s > 3$ an integration by parts shows that

$$\begin{aligned}
\int_2^T t\,e^{-(s-3)t}\,\mathrm{d}t &= -t\,\frac{e^{-(s-3)t}}{s-3}\bigg|_2^T + \int_2^T \frac{e^{-(s-3)t}}{s-3}\,\mathrm{d}t \\
&= \left(-t\,\frac{e^{-(s-3)t}}{s-3} - \frac{e^{-(s-3)t}}{(s-3)^2} \right)\bigg|_2^T \\
&= \left(-T\,\frac{e^{-(s-3)T}}{s-3} - \frac{e^{-(s-3)T}}{(s-3)^2} \right) + \left(2\,\frac{e^{-(s-3)2}}{s-3} + \frac{e^{-(s-3)2}}{(s-3)^2} \right).
\end{aligned}$$

Hence, for $s > 3$ we have that

$$\begin{aligned}
\mathcal{L}[f](s) &= \lim_{T\to\infty} \left[\left(-T\,\frac{e^{-(s-3)T}}{s-3} - \frac{e^{-(s-3)T}}{(s-3)^2} \right) + \left(2\,\frac{e^{-(s-3)2}}{s-3} + \frac{e^{-(s-3)2}}{(s-3)^2} \right) \right] \\
&= \frac{e^{-(s-3)2}}{(s-3)^2} + 2\,\frac{e^{-(s-3)2}}{s-3} - \lim_{T\to\infty} \left(T\,\frac{e^{-(s-3)T}}{s-3} + \frac{e^{-(s-3)T}}{(s-3)^2} \right) \\
&= \frac{e^{-(s-3)2}}{(s-3)^2} + 2\,\frac{e^{-(s-3)2}}{s-3}\,.
\end{aligned}$$

(2) Consider the following MATLAB commands.

```
>> syms t y(t) s Y
>> f = heaviside(t)*t^2 + heaviside(t − 3)*(3*t − t^2);
>> diffeqn = diff(y, 2) − 6*diff(y, 1) + 10*y(t) == f;
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, ...
                [laplace(y(t), t, s), y(0), subs(diff(y(t), t), t, 0)], [Y, 2, 3]);
>> ytrans = simplify(solve(algeqn, Y));
>> y = ilaplace(ytrans, s, t)
```

(a) Give the initial-value problem for $y(t)$ that is being solved.
(b) Find the Laplace transform $Y(s)$ of the solution $y(t)$.

DO NOT take the inverse Laplace transform of $Y(s)$ to find $y(t)$, just solve for $Y(s)$! You may refer to the table on the last page.

Solution (a). The initial-value problem for $y(t)$ that is being solved is

$$y'' - 6y' + 10y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = 3\,,$$

where the forcing $f(t)$ can be expressed either as

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 3\,, \\ 3t & \text{for } 3 \le t\,, \end{cases}$$

or in terms of the unit step function as $f(t) = t^2 + u(t-3)(3t - t^2)$.

Solution (b). The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) - 6\mathcal{L}[y'](s) + 10\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = s\,Y(s) - 2\,, \\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2 Y(s) - 2s - 3\,. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, we first write $f(t)$ as

$$f(t) = t^2 + u(t-3)(3t - t^2) = t^2 + u(t-3)j(t-3)\,,$$

where by setting $j(t-3) = 3t - t^2$ we see by the shifty step method that

$$j(t) = 3(t+3) - (t+3)^2 = 3t + 9 - t^2 - 6t - 9 = -t^2 - 3t\,.$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 2$ and with $a = 0$ and $n = 1$ shows that

$$\mathcal{L}[t^2](s) = \frac{2}{s^3}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,,$$

whereby item 6 with $c = 3$ and $j(t) = -t^2 - 3t$ shows that

$$\mathcal{L}\big[u(t-3)j(t-3)\big](s) = e^{-3s}\mathcal{L}[j](s) = -e^{-3s}\mathcal{L}\big[t^2 + 3t\big](s) = -e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore

$$\mathcal{L}[f](s) = \mathcal{L}\big[t^2 + u(t-3)j(t-3)\big](s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2 Y(s) - 2s - 3\big) - 6\big(sY(s) - 2\big) + 10Y(s) = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right),$$

which becomes

$$(s^2 - 6s + 10)Y(s) - 2s + 9 = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right).$$

Therefore $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 - 6s + 10}\left(2s - 9 + \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right)\right).$$

(3) Find $Y(s) = \mathcal{L}[y](s)$ where $y(t)$ solves the initial-value problem

$$y'' + 4y' + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \le t < 2\pi\,, \\ t - 2\pi & \text{for } t \ge 2\pi\,. \end{cases}$$

DO NOT take the inverse Laplace transform of $Y(s)$ to find $y(t)$, just solve for $Y(s)$! You may refer to the table on the last page.

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 13\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= s\,\mathcal{L}[y](s) - y(0) = s\,Y(s) - 4\,, \\ \mathcal{L}[y''](s) &= s\,\mathcal{L}[y'](s) - y'(0) = s^2Y(s) - 4s - 1\,. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned} f(t) &= \big(1 - u(t-2\pi)\big)\cos(t) + u(t-2\pi)(t-2\pi) \\ &= \cos(t) + u(t-2\pi)\big(t - 2\pi - \cos(t)\big) \\ &= \cos(t) + u(t-2\pi)j(t-2\pi)\,, \end{aligned}$$

where by setting $j(t-2\pi) = t - 2\pi - \cos(t)$ we see by the shifty step method that

$$j(t) = (t + 2\pi) - 2\pi - \cos(t + 2\pi) = t - \cos(t)\,.$$

Here we have used the fact that $\cos(t)$ is 2π-periodic. Referring to the table on the last page, item 6 with $c = 2\pi$ shows that

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}[\cos(t)](s) + \mathcal{L}[u(t-2\pi)j(t-2\pi)](s) \\ &= \mathcal{L}[\cos(t)](s) + e^{-2\pi s}\mathcal{L}[j(t)](s) \\ &= \mathcal{L}[\cos(t)](s) + e^{-2\pi s}\mathcal{L}[t - \cos(t)](s)\,. \end{aligned}$$

Then item 2 with $a = 0$ and $b = 1$, and item 1 with $n = 1$ and $a = 1$ imply that

$$\mathcal{L}[f](s) = \frac{s}{s^2+1} + e^{-2\pi s}\left(\frac{1}{s^2} - \frac{s}{s^2+1}\right)\,.$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 4s - 1\big) + 4\big(sY(s) - 4\big) + 13Y(s) = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,,$$

which becomes

$$(s^2 + 4s + 13)Y(s) - 4s - 1 - 16 = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 4s + 13}\left(4s + 17 + \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\right)\,.$$

(4) Find $X(s) = \mathcal{L}[x](s)$ where $x(t)$ solves the initial-value problem

$$x'' + 4x = \delta(t-3)\,, \qquad x(0) = 5\,, \quad x'(0) = 0\,.$$

DO NOT take the inverse Laplace transform of $X(s)$ to find $x(t)$, just solve for $X(s)$! You may refer to the table on the last page.

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[x''](s) + 4\mathcal{L}[x](s) = \mathcal{L}[\delta(t-3)](s)\,,$$

where

$$\begin{aligned}\mathcal{L}[x](s) &= X(s)\,,\\ \mathcal{L}[x'](s) &= s\,\mathcal{L}[x](s) - x(0) = s\,X(s) - 5\,,\\ \mathcal{L}[x''](s) &= s\,\mathcal{L}[x'](s) - x'(0) = s^2 X(s) - 5s - 0\,.\end{aligned}$$

Referring to the table on the last page, item 7 with $c = 3$ and $h(t) = 1$ shows that

$$\mathcal{L}[\delta(t-3)](s) = e^{-3s}\,.$$

The Laplace transform of the initial-value problem then becomes

$$(s^2+4)X(s) - 5s = e^{-3s}\,.$$

Hence, $X(s)$ is given by

$$X(s) = \frac{5s + e^{-3s}}{s^2+4}\,.$$

Remark. You should be able to take the inverse Laplace transform to obtain

$$x(t) = \mathcal{L}^{-1}[X](t) = \mathcal{L}^{-1}\left[\frac{5s+e^{-3s}}{s^2+4}\right](t) = 5\cos(2t) + \tfrac{1}{2}u(t-3)\sin(2(t-3))\,.$$

(5) Find the inverse Laplace transforms of the following functions.

(a) $F(s) = \dfrac{2}{(s+5)^2}$,

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}$,

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}$.

You may refer to the table on the last page.

Solution (a). Referring to the table on the last page, item 1 with $n = 1$ and $a = -5$ gives

$$\mathcal{L}[t\,e^{-5t}](s) = \frac{1}{(s+5)^2}\,.$$

Therefore we conclude that

$$\mathcal{L}^{-1}\left[\frac{2}{(s+5)^2}\right](t) = 2\mathcal{L}^{-1}\left[\frac{1}{(s+5)^2}\right](t) = 2t\,e^{-5t}\,.$$

Solution (b). Because the denominator factors as $(s-3)(s+2)$, we have the partial fraction identity

$$\frac{3s}{s^2-s-6} = \frac{3s}{(s-3)(s+2)} = \frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}\,.$$

Referring to the table on the last page, item 1 with $n=0$ and $a=3$, and with $n=0$ and $a=-2$ gives

$$\mathcal{L}[e^{3t}](s) = \frac{1}{s-3}\,, \qquad \mathcal{L}[e^{-2t}](s) = \frac{1}{s+2}\,.$$

Therefore we conclude that

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{3s}{s^2-s-6}\right](t) &= \mathcal{L}^{-1}\left[\frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}\right](t) \\ &= \tfrac{9}{5}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right](t) + \tfrac{6}{5}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t) \\ &= \tfrac{9}{5}e^{3t} + \tfrac{6}{5}e^{-2t}\,.\end{aligned}$$

Solution (c). Complete the square in the denominator to get $(s-2)^2+1$. Referring to the table on the last page, item 2 with $a=2$ and $b=1$ gives

$$\mathcal{L}[e^{2t}\cos(t)](s) = \frac{s-2}{(s-2)^2+1}\,.$$

Item 6 with $c=3$ and $j(t)=e^{2t}\cos(t)$ then gives

$$\mathcal{L}[u(t-3)e^{2(t-3)}\cos(t-3)](s) = e^{-3s}\frac{s-2}{(s-2)^2+1}\,.$$

Therefore we conclude that

$$\mathcal{L}^{-1}\left[e^{-3s}\frac{s-2}{s^2-4s+5}\right](t) = u(t-3)e^{2(t-3)}\cos(t-3)\,.$$

(6) For each of the following differential operators compute its Green function $g(t)$ and its natural fundamental set for $t=0$.

(a) $\mathrm{L} = \mathrm{D}^4 + 8\mathrm{D}^2 - 9\,,$

(b) $\mathrm{L} = (\mathrm{D}-2)^3\,.$

You may refer to the table on the last page.

Solution (a). The characteristic polynomial of $\mathrm{L} = \mathrm{D}^4+8\mathrm{D}^2-9$ is $p(s) = s^4+8s^2-9$. Therefore its Green function $g(t)$ is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s^4+8s^2-9}\right](t)\,.$$

Because $p(s)$ factors as $p(s) = (s^2-1)(s^2+9)$ we have the partial fraction identity

$$\frac{1}{s^4+8s^2-9} = \frac{1}{(s^2-1)(s^2+9)} = \frac{\frac{1}{10}}{s^2-1} + \frac{-\frac{1}{10}}{s^2+9}\,.$$

Because s^2-1 factors as $s^2-1=(s-1)(s+1)$ we have the partial fraction identity

$$\frac{1}{s^2-1}=\frac{1}{(s-1)(s+1)}=\frac{\frac{1}{2}}{s-1}+\frac{-\frac{1}{2}}{s+1}\,.$$

By combining the above partial fraction identities we obtain

$$\frac{1}{s^4+8s^2-9}=\tfrac{1}{20}\frac{1}{s-1}-\tfrac{1}{20}\frac{1}{s+1}-\tfrac{1}{10}\frac{1}{s^2+9}\,.$$

Referring to the table on the last page, item 1 with $a=1$ and $n=0$ and with $a=-1$ and $n=0$ gives

$$\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t)=e^t\,,\qquad \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t)=e^{-t}\,,$$

while item 3 with $a=0$ and $b=3$ gives

$$\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right](t)=\sin(3t)\,.$$

Therefore the Green function $g(t)$ is given by

$$\begin{aligned}
g(t)&=\mathcal{L}^{-1}\left[\frac{1}{s^4+8s^2-9}\right](t)\\
&=\tfrac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t)-\tfrac{1}{20}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t)-\tfrac{1}{30}\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right](t)\\
&=\tfrac{1}{20}e^t-\tfrac{1}{20}e^{-t}-\tfrac{1}{30}\sin(3t)\,.
\end{aligned}$$

Then because we see the characteristic polynomial as

$$p(s)=s^4+0s^3+8s^2+0s-9\,,$$

the natrual fundamental set for $t=0$ is found by

$$\begin{aligned}
N_3(t)&=g(t)=\tfrac{1}{20}e^t-\tfrac{1}{20}e^{-t}-\tfrac{1}{30}\sin(3t)\,,\\
N_2(t)&=N_3'(t)+0g(t)=\tfrac{1}{20}e^t+\tfrac{1}{20}e^{-t}-\tfrac{1}{10}\cos(3t)\,,\\
N_1(t)&=N_2'(t)+8g(t)\\
&=\tfrac{1}{20}e^t-\tfrac{1}{20}e^{-t}+\tfrac{3}{10}\sin(3t)+\tfrac{8}{20}e^t-\tfrac{8}{20}e^{-t}-\tfrac{8}{30}\sin(3t)\,,\\
&=\tfrac{9}{20}e^t-\tfrac{9}{20}e^{-t}+\tfrac{1}{30}\sin(3t)\,,\\
N_0(t)&=N_1'(t)+0g(t)=\tfrac{9}{20}e^t+\tfrac{9}{20}e^{-t}+\tfrac{1}{10}\cos(3t)\,.
\end{aligned}$$

Remark. The calculation of the natural fundamental set is a bit simpler if the Green function is expressed in terms of hyperbolic functions. It becomes

$$\begin{aligned}
N_3(t)&=g(t)=\tfrac{1}{10}\sinh(t)-\tfrac{1}{30}\sin(3t)\,,\\
N_2(t)&=N_3'(t)+0g(t)=\tfrac{1}{10}\cosh(t)-\tfrac{1}{10}\cos(3t)\,,\\
N_1(t)&=N_2'(t)+8g(t)\\
&=\tfrac{1}{10}\sinh(t)+\tfrac{3}{10}\sin(3t)+\tfrac{8}{10}\sinh(t)-\tfrac{8}{30}\sin(3t)\,,\\
&=\tfrac{9}{10}\sinh(t)+\tfrac{1}{30}\sin(3t)\,,\\
N_0(t)&=N_1'(t)+0g(t)=\tfrac{9}{10}\cosh(t)+\tfrac{1}{10}\cos(3t)\,.
\end{aligned}$$

Solution (b). The characteristic polynomial of $\mathrm{L} = (\mathrm{D} - 2)^3$ is $p(s) = (s-2)^3$. Therefore its Green function $g(t)$ is given by

$$g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s-2)^3}\right](t)\,.$$

Referring to the table on the last page, item 1 with $a = 2$ and $n = 2$ gives

$$g(t) = \tfrac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s-2)^3}\right](t) = \tfrac{1}{2}t^2 e^{2t}\,.$$

Then because by the binomial expansion we see the characteristic polynomial as

$$\begin{aligned} p(s) = (s-2)^3 &= s^3 + 3(-2)s^2 + 3(-2)^2 s + (-2)^3 \\ &= s^3 - 6s^2 + 12s - 8\,, \end{aligned}$$

the natrual fundamental set for $t = 0$ is found by

$$\begin{aligned} N_2(t) &= g(t) = \tfrac{1}{2}t^2 e^{2t}\,, \\ N_1(t) &= N_2'(t) - 6g(t) \\ &= \left(t\,e^{2t} + t^2 e^{2t}\right) - \tfrac{6}{2}t^2 e^{2t} = t\,e^{2t} - 2t^2 e^{2t}\,, \\ N_0(t) &= N_1'(t) + 12g(t) \\ &= \left(e^{2t} - 2t\,e^{2t} - 4t^2 e^{2t}\right) + \tfrac{12}{2}t^2 e^{2t} = e^{2t} - 2t\,e^{2t} + 2t^2 e^{2t}\,. \end{aligned}$$

(7) Recast the equation $u''' + t^2 u' - 3u = \sinh(2t)$ as a first-order system of ordinary differential equations.

Solution. Because the equation is third order, the first-order system must have dimension three. The simplest such first-order system is

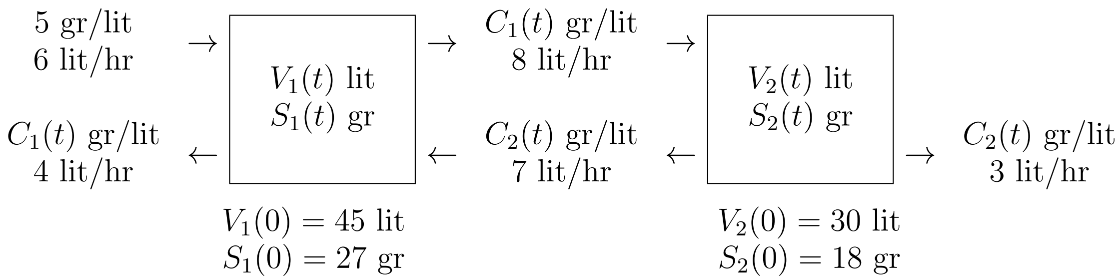
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \sinh(2t) + 3x_1 - t^2 x_2 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}.$$

(8) Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 45 liters and the second contains 30 liters. Brine with a salt concentration of 5 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 8 liters per hour, from the second into the first at 7 liters per hour, from the first into a drain at 4 liter per hour, and from the second into a drain at 3 liters per hour. At $t = 0$ there are 27 grams of salt in the first tank and 18 grams in the second.

(a) Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(b) Give the interval of definition for the solution of this initial-value problem.

Solution (a). Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t hours. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t hours. Because mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the concentrations of the brine that flows out of these tanks. We have the following picture.



We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will be $V_1(t) = 45 + t$ liters of brine in the first tank and $V_2(t) = 30 - 2t$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem

$$\begin{aligned}\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 5\cdot 6 + \frac{S_2}{30-2t}\,7 - \frac{S_1}{45+t}\,8 - \frac{S_1}{45+t}\,4\,, && S_1(0) = 27\,,\\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{45+t}\,8 - \frac{S_2}{30-2t}\,7 - \frac{S_2}{30-2t}\,3\,, && S_2(0) = 18\,.\end{aligned}$$

You could leave the answer in the above form. However, it can be simplified to

$$\begin{aligned}\frac{\mathrm{d}S_1}{\mathrm{d}t} &= 30 + \frac{7}{30-2t}\,S_2 - \frac{12}{45+t}\,S_1\,, && S_1(0) = 27\,,\\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{8}{45+t}\,S_1 - \frac{5}{15-t}\,S_2\,, && S_2(0) = 18\,.\end{aligned}$$

Solution (b). This first-order system of differential equations is *linear*.

- ⋄ Its coefficients are undefined either at $t = -45$ or at $t = 15$ and are continuous elsewhere.
- ⋄ Its forcing is constant, so is continuous everywhere.
- ⋄ Its initial time is $t = 0$.

Therefore the natural interval of definition for the solution of this initial-value problem is $(-45, 15)$ because:

- the initial time $t = 0$ is in $(-45, 15)$;
- all the coefficients and the forcing are continuous over $(-45, 15)$;
- two coefficients are undefined at $t = -45$;
- two coefficients are undefined at $t = 15$.

However, it could also be argued that the interval of definition for the solution of this initial-value problem is $[0, 15)$ because the word problem starts at $t = 0$.

(9) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices
(a) $\mathbf{A}^{\mathrm{T}}$,
(b) $\overline{\mathbf{A}}$,
(c) $\mathbf{A}^{\mathrm{H}}$,
(d) $5\mathbf{A} - \mathbf{B}$,
(e) $\mathbf{AB}$,
(f) $\mathbf{B}^{-1}$.

Solution (a). The transpose of $\mathbf{A}$ is

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} -i2 & 2+i \\ 1+i & -4 \end{pmatrix}.$$

Solution (b). The conjugate of $\mathbf{A}$ is

$$\overline{\mathbf{A}} = \begin{pmatrix} i2 & 1-i \\ 2-i & -4 \end{pmatrix}.$$

Solution (c). The Hermitian transpose of $\mathbf{A}$ is

$$\mathbf{A}^{\mathrm{H}} = \begin{pmatrix} i2 & 2-i \\ 1-i & -4 \end{pmatrix}.$$

Solution (d). The difference of $5\mathbf{A}$ and $\mathbf{B}$ is given by

$$5\mathbf{A} - \mathbf{B} = \begin{pmatrix} -i10 & 5+i5 \\ 10+i5 & -20 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} -7-i10 & -1+i5 \\ 2+i5 & -27 \end{pmatrix}.$$

Solution (e). The product of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -i2\cdot 7 + (1+i)\cdot 8 & -i2\cdot 6 + (1+i)\cdot 7 \\ (2+i)\cdot 7 - 4\cdot 8 & (2+i)\cdot 6 - 4\cdot 7 \end{pmatrix} \\ &= \begin{pmatrix} 8-i6 & 7-i5 \\ -18+i7 & -16+i6 \end{pmatrix}. \end{aligned}$$

Solution (f). Observe that it is clear that $\mathbf{B}$ has an inverse because

$$\det(\mathbf{B}) = \det\begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = 7\cdot 7 - 6\cdot 8 = 49 - 48 = 1\,.$$

Then the inverse of $\mathbf{B}$ is given by

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix}.$$

(10) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) Give a fundamental matrix $\boldsymbol{\Psi}(t)$ for the system found in part (b).

(d) For the system found in part (b), solve the initial-value problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x}\,, \qquad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(e) For the $\mathbf{A}(t)$ found in part (b), give the Green matrix for the system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,.$$

Solution (a).

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} = 3t^4 + 9 - 2t^4 = t^4 + 9\,.$$

Solution (b). Let $\boldsymbol{\Psi}(t) = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}$. Because $\dfrac{d\boldsymbol{\Psi}(t)}{dt} = \mathbf{A}(t)\boldsymbol{\Psi}(t)$, we have

$$\begin{aligned} \mathbf{A}(t) &= \frac{d\boldsymbol{\Psi}(t)}{dt}\boldsymbol{\Psi}(t)^{-1} = \begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix} \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}^{-1} \\ &= \frac{1}{t^4+9}\begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix} \begin{pmatrix} 3 & -t^2 \\ -2t^2 & t^4+3 \end{pmatrix} = \frac{1}{t^4+9}\begin{pmatrix} 8t^3 & 6t - 2t^5 \\ 12t & -4t^3 \end{pmatrix}. \end{aligned}$$

Solution (c). Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a fundamental matrix for the system found in part (b) is simply given by

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}.$$

Solution (d). Because a fundamental matrix $\boldsymbol{\Psi}(t)$ for the system found in part (b) was given in part (c), the solution of the initial-value problem is

$$\begin{aligned} \mathbf{x}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}(1)^{-1}\mathbf{x}(1) &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \frac{1}{10} \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3t^4 + 9 - 2t^2 \\ 6t^2 - 6 \end{pmatrix}. \end{aligned}$$

Alternative Solution (d). Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 3 \end{pmatrix} .$$

The initial condition then implies that

$$\mathbf{x}(1) = c_1 \begin{pmatrix} 4 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4c_1 + c_2 \\ 2c_1 + 3c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

from which we see that $c_1 = \frac{3}{10}$ and $c_2 = -\frac{1}{5}$. The solution of the initial-value problem is thereby

$$\mathbf{x}(t) = \tfrac{3}{10} \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix} - \tfrac{1}{5} \begin{pmatrix} t^2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}t^4 - \frac{1}{5}t^2 + \frac{9}{10} \\ \frac{3}{5}t^2 - \frac{3}{5} \end{pmatrix} .$$

Solution (e). Because a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b) was given in part (c), the Green matrix for the nonhomogeneous system is

$$\begin{aligned} \mathbf{G}(t,s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \begin{pmatrix} s^4+3 & s^2 \\ 2s^2 & 3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} \frac{1}{s^4+9} \begin{pmatrix} 3 & -s^2 \\ -2s^2 & s^4+3 \end{pmatrix} \\ &= \frac{1}{s^4+9} \begin{pmatrix} 3t^4 + 9 - 2t^2s^2 & t^2(s^4+3) - (t^4+3)s^2 \\ 6t^2 - 6s^2 & 3s^4 + 9 - 2t^2s^2 \end{pmatrix} . \end{aligned}$$

(11) Compute $e^{t\mathbf{A}}$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

Solution (a) by Two-by-Two Formula. Because

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} ,$$

the characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z-1)^2 - 2^2 .$$

This is a difference of squares with $\mu = 1$ and $\nu = 2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cosh(2t)\mathbf{I} + \frac{\sinh(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[\cosh(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(2t)}{2} \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix} . \end{aligned}$$

Solution (a) by the Natural Fundamental Set Method. Because

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix},$$

the characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z+1)(z-3).$$

Below we show that the natural fundamental set of solutions for $t = 0$ associated with $p(\mathrm{D}) = \mathrm{D}^2 - 2\mathrm{D} + 3$ is

$$N_0(t) = \frac{e^{3t} + 3e^{-t}}{4}, \qquad N_1(t) = \frac{e^{3t} - e^{-t}}{4}.$$

Then

$$\begin{aligned} e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A} &= \frac{e^{3t} + 3e^{-t}}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{3t} - e^{-t}}{4} \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{3t} + 2e^{-t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}. \end{aligned}$$

From the Green Function. By the partial fraction identity

$$\frac{1}{s^2 - 2s + 3} = \frac{1}{(s-3)(s+1)} = \frac{\frac{1}{4}}{s-3} + \frac{-\frac{1}{4}}{s+1},$$

the Green function associated with $p(\mathrm{D}) = \mathrm{D}^2 - 2\mathrm{D} + 3$ is

$$\begin{aligned} g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2 - 2s + 3}\right](t) \\ &= \tfrac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right](t) - \tfrac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) = \tfrac{1}{4}e^{3t} - \tfrac{1}{4}e^{-t}. \end{aligned}$$

Then, because the characteristic polynomial is $p(s) = s^2 - 2s + 3$, the natural fundamental set is

$$\begin{aligned} N_1(t) &= g(t) = \tfrac{1}{4}e^{3t} - \tfrac{1}{4}e^{-t}, \\ N_0(t) &= N_1'(t) - 2g(t) = \left(\tfrac{3}{4}e^{3t} + \tfrac{1}{4}e^{-t}\right) - \left(\tfrac{2}{4}e^{3t} - \tfrac{2}{4}e^{-t}\right) = \tfrac{1}{4}e^{3t} + \tfrac{3}{4}e^{-t}. \end{aligned}$$

From the General Initial-Value Problem. The general initial-value problem associated with $p(\mathrm{D}) = \mathrm{D}^2 - 2\mathrm{D} + 3$ is

$$y'' - 2y' - 3y = 0, \qquad y(0) = y_0, \quad y'(0) = y_1.$$

This has the general solution $y(t) = c_1 e^{3t} + c_2 e^{-t}$. Because $y'(t) = 3c_1 e^{3t} - c_2 e^{-t}$, the general initial conditions yield

$$y_0 = y(0) = c_1 + c_2, \qquad y_1 = y'(0) = 3c_1 - c_2.$$

This system can be solved to obtain

$$c_1 = \frac{y_0 + y_1}{4}, \qquad c_2 = \frac{3y_0 - y_1}{4}.$$

The solution of the general initial-value problem is thereby

$$y(t) = \frac{y_0 + y_1}{4}\, e^{3t} + \frac{3y_0 - y_1}{4}\, e^{-t} = \frac{e^{3t} + 3e^{-t}}{4}\, y_0 + \frac{e^{3t} - e^{-t}}{4}\, y_1.$$

Therefore the associated natural fundamental set of solutions is
$$N_0(t) = \frac{e^{3t} + 3e^{-t}}{4}\,, \qquad N_1(t) = \frac{e^{3t} - e^{-t}}{4}\,.$$

Solution (a) by Eigen Methods. Because
$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\,,$$
the characteristic polynomial of $\mathbf{A}$ is
$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z+1)(z-3)\,.$$
The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -1 and 3. Because
$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\,, \qquad \mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\,,$$
we can read off that $\mathbf{A}$ has the eigenpairs
$$\left(-1\,, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)\,, \qquad \left(3\,, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)\,.$$
Set
$$\mathbf{V} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}\,, \qquad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\,.$$
Because $\det(\mathbf{V}) = 4$, we see that
$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & -2e^{-t} \\ e^{3t} & 2e^{3t} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2e^{-t} + 2e^{3t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{-t} + 2e^{3t} \end{pmatrix}\,. \end{aligned}$$

Solution (b) by Two-by-Two Formula. Because
$$\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}\,,$$
the characteristic polynomial of $\mathbf{A}$ is given by
$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z-4)^2\,.$$
This is a perfect square with $\mu = 4$. Hence,
$$\begin{aligned} e^{t\mathbf{A}} = e^{4t}\Big[\mathbf{I} + t\,(\mathbf{A} - 4\mathbf{I})\Big] &= e^{4t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}\right] \\ &= e^{4t} \begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix}\,. \end{aligned}$$

Solution (b) by the Natural Fundamental Set Method. Because
$$\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}\,,$$

the characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z-4)^4 .$$

Below we show that the natural fundamental set of solutions for $t = 0$ associated with $p(\mathrm{D}) = \mathrm{D}^2 - 8\mathrm{D} + 16$ is

$$N_0(t) = (1-4t)e^{4t} , \qquad N_1(t) = t\,e^{4t} .$$

Then

$$\begin{aligned} e^{t\mathbf{A}} &= N_0(t)\mathbf{I} + N_1(t)\mathbf{A} = (1-4t)e^{4t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\,e^{4t}\begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix} \\ &= e^{4t}\begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix} . \end{aligned}$$

From the Green Function. Because $p(s) = s^2 - 8s + 16 = (s-4)^2$, the Green function associated with $p(\mathrm{D}) = \mathrm{D}^2 - 8\mathrm{D} + 16$ is

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 - 8s + 16}\right](t) \\ &= \mathcal{L}^{-1}\left[\frac{1}{(s-4)^2}\right](t) = t\,e^{4t} . \end{aligned}$$

Then, because the characteristic polynomial is $p(s) = s^2 - 8s + 16$, the natural fundamental set is

$$\begin{aligned} &N_1(t) = g(t) = t\,e^{4t} , \\ &N_0(t) = N_1'(t) - 8g(t) = \big(e^{4t} + 4t\,e^{4t}\big) - 8t\,e^{4t} = e^{4t} - 4t\,e^{4t} . \end{aligned}$$

From the General Initial-Value Problem. The general initial-value problem associated with $p(\mathrm{D}) = \mathrm{D}^2 - 8\mathrm{D} + 16$ is

$$y'' - 8y' + 16y = 0 , \qquad y(0) = y_0 , \quad y'(0) = y_1 .$$

This has the general solution $y(t) = c_1 e^{4t} + c_2 t e^{4t}$. Because

$$y'(t) = 4c_1 e^{3t} + 4c_2 t e^{-t} + c_2 e^{4t} ,$$

the general initial conditions yield

$$y_0 = y(0) = c_1 , \qquad y_1 = y'(0) = 4c_1 + c_2 .$$

This system can be solved to obtain $c_1 = y_0$ and $c_2 = y_1 - 4y_0$. The solution of the general initial-value problem is thereby

$$y(t) = y_0\,e^{4t} + (y_1 - 4y_0)t\,e^{4t} = (1-4t)e^{4t}\,y_0 + t\,e^{4t}\,y_1 .$$

Therefore the associated natural fundamental set of solutions is

$$N_0(t) = (1-4t)e^{4t} , \qquad N_1(t) = t\,e^{4t} .$$

(12) Give the Green matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$ when

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

Solution (a). By the solution to part (a) of the previous problem

$$e^{t\mathbf{A}} = \frac{1}{4}\begin{pmatrix} 2e^{3t} + 2e^{-t} & 4e^{3t} - 4e^{-t} \\ e^{3t} - e^{-t} & 2e^{3t} + 2e^{-t} \end{pmatrix}.$$

Therefore the Green matrix $\mathbf{G}(t,s)$ is given by

$$\mathbf{G}(t,s) = e^{t\mathbf{A}}e^{-s\mathbf{A}} = e^{(t-s)\mathbf{A}} = \frac{1}{4}\begin{pmatrix} 2e^{3(t-s)} + 2e^{-(t-s)} & 4e^{3(t-s)} - 4e^{-(t-s)} \\ e^{3(t-s)} - e^{-(t-s)} & 2e^{3(t-s)} + 2e^{-(t-s)} \end{pmatrix}.$$

Solution (b). By the solution to part (b) of the previous problem

$$e^{t\mathbf{A}} = e^{4t}\begin{pmatrix} 1 + 2t & 4t \\ -t & 1 - 2t \end{pmatrix}.$$

Therefore the Green matrix $\mathbf{G}(t,s)$ is given by

$$\mathbf{G}(t,s) = e^{t\mathbf{A}}e^{-s\mathbf{A}} = e^{(t-s)\mathbf{A}} = e^{4(t-s)}\begin{pmatrix} 1 + 2(t-s) & 4(t-s) \\ -(t-s) & 1 - 2(t-s) \end{pmatrix}.$$

(13) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix}.$$

Compute $e^{t\mathbf{A}}$ given that the characteristic polynomial of $\mathbf{A}$ is $p(z) = z^3 + 9z$ and that the natural fundamental set of solutions associated with $t = 0$ for $\mathrm{D}^3 + 9\mathrm{D}$ is

$$N_0(t) = 1\,, \qquad N_1(t) = \tfrac{1}{3}\sin(3t)\,, \qquad N_2(t) = \tfrac{1}{9}\big(1 - \cos(3t)\big)\,.$$

Solution. The natural fundamental set method says that

$$e^{t\mathbf{A}} = N_0(t)\mathbf{I} + N_1(t)\mathbf{A} + N_2(t)\mathbf{A}^2\,.$$

Because $N_0(t) = 1$, $N_1(t) = \frac{1}{3}\sin(3t)$, $N_2(t) = \frac{1}{9}\big(1 - \cos(3t)\big)$, and

$$\begin{aligned}\mathbf{A}^2 &= \begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -2 & 1 \\ 4 & 0 & -2 \\ -2 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - 8 - 2 & 2 & -1 + 4 + 1 \\ -4 + 0 + 4 & -8 & 4 - 0 - 2 \\ 2 + 0 - 2 & 4 & -2 - 0 + 1 \end{pmatrix} = \begin{pmatrix} -9 & 2 & 4 \\ 0 & -8 & 2 \\ 0 & 4 & -1 \end{pmatrix},\end{aligned}$$

we see that

$$e^{t\mathbf{A}} = 1\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \tfrac{1}{3}\sin(3t)\begin{pmatrix}-1&-2&1\\4&0&-2\\-2&0&1\end{pmatrix} + \tfrac{1}{9}\big(1-\cos(3t)\big)\begin{pmatrix}-9&2&4\\0&-8&2\\0&4&-1\end{pmatrix}$$
$$= \begin{pmatrix}\cos(3t)-\frac{1}{3}\sin(3t) & -\frac{2}{3}\sin(3t)+\frac{2}{3}-\frac{2}{3}\cos(3t) & \frac{1}{3}\sin(3t)+\frac{4}{9}-\frac{4}{9}\cos(3t)\\ \frac{4}{3}\sin(3t) & \frac{1}{9}+\frac{8}{9}\cos(3t) & -\frac{2}{3}\sin(3t)+\frac{2}{9}-\frac{2}{9}\cos(3t)\\ -\frac{2}{3}\sin(3t) & \frac{4}{9}-\frac{4}{9}\cos(3t) & \frac{8}{9}+\frac{1}{9}\cos(3t)+\frac{1}{3}\sin(3t)\end{pmatrix}.$$

(14) Solve each of the following initial-value problems.

(a) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2&2\\5&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad \begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}.$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1&1\\-4&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \qquad \begin{pmatrix}x(0)\\y(0)\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

Solution (a). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}2&2\\5&-1\end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - z - 12 = (z-\tfrac{1}{2})^2 - (\tfrac{7}{2})^2.$$

This is a difference of squares with $\mu = \frac{1}{2}$ and $\nu = \frac{7}{2}$. Hence,

$$e^{t\mathbf{A}} = e^{\frac{1}{2}t}\left[\cosh\!\big(\tfrac{7}{2}t\big)\mathbf{I} + \frac{\sinh\big(\frac{7}{2}t\big)}{\frac{7}{2}}\big(\mathbf{A}-\tfrac{1}{2}\mathbf{I}\big)\right]$$
$$= e^{\frac{1}{2}t}\left[\cosh\!\big(\tfrac{7}{2}t\big)\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{\sinh\big(\frac{7}{2}t\big)}{\frac{7}{2}}\begin{pmatrix}\frac{3}{2}&2\\5&-\frac{3}{2}\end{pmatrix}\right]$$
$$= e^{\frac{1}{2}t}\begin{pmatrix}\cosh\big(\frac{7}{2}t\big)+\frac{3}{7}\sinh\big(\frac{7}{2}t\big) & \frac{4}{7}\sinh\big(\frac{7}{2}t\big)\\ \frac{10}{7}\sinh\big(\frac{7}{2}t\big) & \cosh\big(\frac{7}{2}t\big)-\frac{3}{7}\sinh\big(\frac{7}{2}t\big)\end{pmatrix}.$$

Therefore the solution of the initial-value problem is

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix}x(0)\\y(0)\end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix}1\\-1\end{pmatrix}$$
$$= e^{\frac{1}{2}t}\begin{pmatrix}\cosh\big(\frac{7}{2}t\big)+\frac{3}{7}\sinh\big(\frac{7}{2}t\big) & \frac{4}{7}\sinh\big(\frac{7}{2}t\big)\\ \frac{10}{7}\sinh\big(\frac{7}{2}t\big) & \cosh\big(\frac{7}{2}t\big)-\frac{3}{7}\sinh\big(\frac{7}{2}t\big)\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}$$
$$= e^{\frac{1}{2}t}\begin{pmatrix}\cosh\big(\frac{7}{2}t\big)-\frac{1}{7}\sinh\big(\frac{7}{2}t\big)\\ -\cosh\big(\frac{7}{2}t\big)+\frac{13}{7}\sinh\big(\frac{7}{2}t\big)\end{pmatrix}.$$

Solution (b). The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}1&1\\-4&1\end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 5 = (z-1)^2 + 2^2.$$

This is a sum of squares with $\mu = 1$ and $\nu = 2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}(\mathbf{A} - \mathbf{I}) \right] \\ &= e^t \left[\cos(2t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix} . \end{aligned}$$

Therefore the solution of the initial-value problem is

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^t \begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) \\ -2\sin(2t) + \cos(2t) \end{pmatrix} . \end{aligned}$$

Remark. We could have used other methods to compute $e^{t\mathbf{A}}$ in each part of the above problem. Alternatively, we could have constructed a fundamental matrix $\mathbf{\Psi}(t)$ and then determined $\mathbf{c}$ so that $\mathbf{\Psi}(t)\mathbf{c}$ satisfies the initial conditions.

(15) Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution (a). We must find a general solution for the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 1 = (z-1)^2 .$$

This is a perfect square with $\mu = 2$. Hence,

$$\begin{aligned} e^{t\mathbf{A}} = e^t \big[\mathbf{I} + t\,(\mathbf{A} - \mathbf{I})\big] &= e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \right] \\ &= e^t \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} . \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= c_1 e^t \begin{pmatrix} 1+2t \\ t \end{pmatrix} + c_2 e^t \begin{pmatrix} -4t \\ 1-2t \end{pmatrix} .$$

Solution (b) by Two-by-Two Formula. We must find a general solution for the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4^2 .$$

This is a sum of squares with $\mu = 0$ and $\nu = 4$. Hence,

$$e^{t\mathbf{A}} = \left[\cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}\mathbf{A} \right] = \left[\cos(4t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(4t)}{4} \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \right]$$
$$= \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} .$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= c_1 \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ \sin(4t) \end{pmatrix} + c_2 \begin{pmatrix} -\frac{5}{4}\sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} .$$

Solution (b) by Eigen Methods. We must find a general solution for the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $\pm i4$. Because

$$\mathbf{A} - i4\mathbf{I} = \begin{pmatrix} 2 - i4 & -5 \\ 4 & -2 - i4 \end{pmatrix} , \qquad \mathbf{A} + i4\mathbf{I} = \begin{pmatrix} 2 + i4 & -5 \\ 4 & -2 + i4 \end{pmatrix} ,$$

we can read off that $\mathbf{A}$ has the eigenpairs

$$\left(i4 \,, \begin{pmatrix} 1 + i2 \\ 2 \end{pmatrix} \right) , \qquad \left(-i4 \,, \begin{pmatrix} 1 - i2 \\ 2 \end{pmatrix} \right) .$$

Therefore the system has the complex-valued solution

$$
\begin{aligned}
e^{i4t}\begin{pmatrix}1+i2\\ 2\end{pmatrix} &= \big(\cos(4t)+i\sin(4t)\big)\begin{pmatrix}1+i2\\ 2\end{pmatrix}\\
&= \begin{pmatrix}\cos(4t)-2\sin(4t)+i2\cos(4t)+i\sin(4t)\\ 2\cos(4t)+i2\sin(4t)\end{pmatrix}.
\end{aligned}
$$

By taking real and imaginary parts, we obtain the two real solutions

$$
\begin{aligned}
\mathbf{x}_1(t) &= \operatorname{Re}\left(e^{i4t}\begin{pmatrix}1+i2\\ 2\end{pmatrix}\right) = \begin{pmatrix}\cos(4t)-2\sin(4t)\\ 2\cos(4t)\end{pmatrix},\\
\mathbf{x}_2(t) &= \operatorname{Im}\left(e^{i4t}\begin{pmatrix}1+i2\\ 2\end{pmatrix}\right) = \begin{pmatrix}2\cos(4t)+\sin(4t)\\ 2\sin(4t)\end{pmatrix}.
\end{aligned}
$$

Therefore a general solution is

$$
\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = c_1\begin{pmatrix}\cos(4t)-2\sin(4t)\\ 2\cos(4t)\end{pmatrix} + c_2\begin{pmatrix}2\cos(4t)+\sin(4t)\\ 2\sin(4t)\end{pmatrix}.
$$

Solution (c) by Two-by-Two Formula. We must find a general solution for the system

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}5 & 4\\ -5 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.
$$

The characteristic polynomial of $\mathbf{A} = \begin{pmatrix}5 & 4\\ -5 & 1\end{pmatrix}$ is given by

$$
p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z-3)^2 + 4^2.
$$

This is a sum of squares with $\mu = 3$ and $\nu = 4$. Hence,

$$
\begin{aligned}
e^{t\mathbf{A}} &= e^{3t}\left[\cos(4t)\mathbf{I} + \frac{\sin(4t)}{4}(\mathbf{A}-3\mathbf{I})\right]\\
&= e^{3t}\left[\cos(4t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\sin(4t)}{4}\begin{pmatrix}2 & 4\\ -5 & -2\end{pmatrix}\right]\\
&= e^{3t}\begin{pmatrix}\cos(4t)+\frac{1}{2}\sin(4t) & \sin(4t)\\ -\frac{5}{4}\sin(4t) & \cos(4t)-\frac{1}{2}\sin(4t)\end{pmatrix}.
\end{aligned}
$$

Therefore a general solution is

$$
\begin{aligned}
\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix}c_1\\ c_2\end{pmatrix} &= e^{3t}\begin{pmatrix}\cos(4t)+\frac{1}{2}\sin(4t) & \sin(4t)\\ -\frac{5}{4}\sin(4t) & \cos(4t)-\frac{1}{2}\sin(4t)\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}\\
&= c_1e^{3t}\begin{pmatrix}\cos(4t)+\frac{1}{2}\sin(4t)\\ -\frac{5}{4}\sin(4t)\end{pmatrix} + c_2e^{3t}\begin{pmatrix}\sin(4t)\\ \cos(4t)-\frac{1}{2}\sin(4t)\end{pmatrix}.
\end{aligned}
$$

Solution (c) by Eigen Methods. We must find a general solution for the system

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}5 & 4\\ -5 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.
$$

The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z-3)^2 + 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $3 \pm i4$. Because

$$\mathbf{A} - (3+i4)\mathbf{I} = \begin{pmatrix} 2-i4 & 4 \\ -5 & -2-i4 \end{pmatrix}, \quad \mathbf{A} - (3-i4)\mathbf{I} = \begin{pmatrix} 2+i4 & 4 \\ -5 & -2+i4 \end{pmatrix},$$

we can read off that $\mathbf{A}$ has the eigenpairs

$$\left(3+i4\,, \begin{pmatrix} -2 \\ 1-i2 \end{pmatrix}\right), \qquad \left(3-i4\,, \begin{pmatrix} -2 \\ 1+i2 \end{pmatrix}\right).$$

Therefore the system has the complex-valued solution

$$\begin{aligned} e^{(3+i4)t}\begin{pmatrix} -2 \\ 1-i2 \end{pmatrix} &= e^{3t}\big(\cos(4t) + i\sin(4t)\big)\begin{pmatrix} -2 \\ 1-i2 \end{pmatrix} \\ &= e^{3t}\begin{pmatrix} -2\cos(4t) - i2\sin(4t) \\ \cos(4t) + 2\sin(4t) + i\sin(4t) - i2\cos(4t) \end{pmatrix}. \end{aligned}$$

By taking real and imaginary parts, we obtain the two real solutions

$$\begin{aligned} \mathbf{x}_1(t) &= \operatorname{Re}\left(e^{(3+i4)t}\begin{pmatrix} -2 \\ 1-i2 \end{pmatrix}\right) = e^{3t}\begin{pmatrix} -2\cos(4t) \\ \cos(4t) + 2\sin(4t) \end{pmatrix}, \\ \mathbf{x}_2(t) &= \operatorname{Im}\left(e^{(3+i4)t}\begin{pmatrix} -2 \\ 1+i2 \end{pmatrix}\right) = e^{3t}\begin{pmatrix} -2\sin(4t) \\ \sin(4t) - 2\cos(4t) \end{pmatrix}. \end{aligned}$$

Therefore a general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{3t}\begin{pmatrix} -2\cos(4t) \\ \cos(4t) + 2\sin(4t) \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} -2\sin(4t) \\ \sin(4t) - 2\cos(4t) \end{pmatrix}.$$

(16) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 1.

Solution. The eigenvectors of $\mathbf{A}$ associated with 1 are all nonzero vectors $\mathbf{v}$ that satisfy $\mathbf{A}\mathbf{v} = \mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ that satisfy $(\mathbf{A}-\mathbf{I})\mathbf{v} = 0$, which is

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0\,.$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} v_1 - v_2 + \ \ v_3 &= 0\,, \\ v_1 \qquad - \ \ v_3 &= 0\,, \\ - v_2 + 2v_3 &= 0\,. \end{aligned}$$

We may solve this system either by elimination or by row reduction. By either method we find that its general solution is

$$v_1 = \alpha\,, \quad v_2 = 2\alpha\,, \quad v_3 = \alpha\,, \qquad \text{for any constant } \alpha\,.$$

The eigenvectors of $\mathbf{A}$ associated with 1 thereby have the form

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \text{for any nonzero constant } \alpha\,.$$

(17) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}.$$

(a) Find all the eigenvalues of $\mathbf{A}$.
(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.
(c) Diagonalize $\mathbf{A}$.
(d) Compute $e^{t\mathbf{A}}$.
(e) Compute $(s\mathbf{I} - \mathbf{A})^{-1}$ for every s where it is defined.

Solution (a). The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 15 = (z-1)^2 - 16\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 1 ± 4, or simply -3 and 5.

Solution (b) by the Cayley-Hamilton Method. We have

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}, \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}.$$

Every nonzero column of $\mathbf{A} - 5\mathbf{I}$ has the form

$$\alpha_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{for some } \alpha_1 \neq 0\,.$$

These are all the eigenvectors associated with -3. Similarly, every nonzero column of $\mathbf{A} + 3\mathbf{I}$ has the form

$$\alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad \text{for some } \alpha_2 \neq 0\,.$$

These are all the eigenvectors associated with 5.

Solution (c). If we use the eigenpairs

$$\left(-3\,, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right), \qquad \left(5\,, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right),$$

then set

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}.$$

Because $\det(\mathbf{V}) = 1\cdot 2 - (-2)\cdot 3 = 2+6 = 8$, we see that

$$\mathbf{V}^{-1} = \frac{1}{8}\begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}.$$

We conclude that $\mathbf{A}$ has the diagonalization

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}\frac{1}{8}\begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}.$$

You do not have to multiply these matrices out. Had we started with different eigenpairs, the steps would be the same as above but we would obtain a different diagonalization.

Solution (d). Because $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ by part (c), we have

$$\begin{aligned} e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1} &= \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{5t} \end{pmatrix}\frac{1}{8}\begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{8}\begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 2e^{-3t} & -3e^{-3t} \\ 2e^{5t} & e^{5t} \end{pmatrix} \\ &= \frac{1}{8}\begin{pmatrix} 2e^{-3t}+6e^{5t} & -3e^{-3t}+3e^{5t} \\ -4e^{-3t}+4e^{5t} & 6e^{-3t}+2e^{5t} \end{pmatrix}. \end{aligned}$$

Solution (e). Because $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ by part (c), we have

$$\begin{aligned} (s\mathbf{I}-\mathbf{A})^{-1} = \mathbf{V}(s\mathbf{I}-\mathbf{D})^{-1}\mathbf{V}^{-1} &= \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s-5} \end{pmatrix}\frac{1}{8}\begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} \\ &= \frac{1}{8}\begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} \frac{2}{s+3} & \frac{-3}{s+3} \\ \frac{2}{s-5} & \frac{1}{s-5} \end{pmatrix} \\ &= \frac{1}{8}\begin{pmatrix} \frac{2}{s+3}+\frac{6}{s-5} & \frac{-3}{s+3}+\frac{3}{s-5} \\ \frac{-4}{s+3}+\frac{4}{s-5} & \frac{6}{s+3}+\frac{2}{s-5} \end{pmatrix}. \end{aligned}$$

This is defined for every s except at $s=-3$ and $s=5$.

(18) What answer will be produced by the following Matlab command?

```
>> A = [1 4; 3 2]; [vect, val] = eig(sym(A))
```

You do not have to give the answer in Matlab format.

Solution. The Matlab command will produce the eigenpairs of $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. The characteristic polynomial of $\mathbf{A}$ is

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 3z - 10 = (z-5)(z+2),$$

so its eigenvalues are 5 and -2. Because

$$\mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}, \qquad \mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix},$$

we can read off that the eigenpairs are

$$\left(5\,, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \qquad \left(-2\,, \begin{pmatrix} -4 \\ 3 \end{pmatrix}\right).$$

(19) A 3×3 matrix $\mathbf{A}$ has the eigenpairs

$$\left(-3\,, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \qquad \left(2\,, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right), \qquad \left(5\,, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right).$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{A}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution (a). One choice for $\mathbf{V}$ and $\mathbf{D}$ is

$$\mathbf{V} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Solution (b). Use the given eigenpairs to construct the special solutions

$$\mathbf{x}_1(t) = e^{-3t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{2t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3(t) = e^{5t}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix},$$

Then a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t) \end{pmatrix} = \begin{pmatrix} e^{-3t} & -e^{2t} & e^{5t} \\ e^{-3t} & e^{2t} & -e^{5t} \\ 0 & e^{2t} & 2e^{5t} \end{pmatrix}.$$

Alternative Solution (b). Given the $\mathbf{V}$ and $\mathbf{D}$ from part (a), a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix} = \begin{pmatrix} e^{-3t} & -e^{2t} & e^{5t} \\ e^{-3t} & e^{2t} & -e^{5t} \\ 0 & e^{2t} & 2e^{5t} \end{pmatrix}.$$

Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[j'(t)](s) = sJ(s) - j(0) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s),\ c \geq 0,$$

and u is the unit step function.

$$\mathcal{L}[\delta(t-c)j(t)](s) = e^{-cs}j(c) \qquad \text{where } c \geq 0 \text{ and } \delta \text{ is the unit impulse.}$$