Sample Problems for the Second In-Class Exam
Math 246, Fall 2019, Professor David Levermore

(1) Give the interval of definition for the solution of the initial-value problem

$$x''' + \frac{\cos(3t)}{4-t}\, x' + \frac{\sin(2t)}{5-t}\, x = \frac{e^{-2t}}{1+t}\,, \qquad x(2) = x'(2) = x''(2) = 0\,.$$

(2) Suppose that $Z_1(t)$, $Z_2(t)$, and $Z_3(t)$ are solutions of the differential equation

$$z''' + 2z'' + (1+t^2)z = 0\,.$$

Suppose we know that $\mathrm{Wr}[Z_1, Z_2, Z_3](1) = 5$. What is $\mathrm{Wr}[Z_1, Z_2, Z_3](t)$?

(3) Show that the functions $X_1(t) = 1$, $X_2(t) = \cos(t)$, and $X_3(t) = \sin(t)$ are linearly independent.

(4) The function $Y(t) = t$ is a solution of the differential equation

$$(t^2 + 4)y'' - 2ty' + 2y = 0\,.$$

Find a real general solution of this equation.

(5) Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-2 + i3$, $-2 - i3$, $i7$, $i7$, $-i7$, $-i7$, 5, 5, 5, -3, 0, 0.

(a) Give the order of L.

(b) Give a real general solution of the homogeneous equation $\mathrm{L}y = 0$.

(6) Give the natural fundamental set of solutions associated with $t = 0$ for each of the following equations.

(a) $v'' - 6v' + 9v = 0$

(b) $\ddot{y} + 4\dot{y} + 20y = 0$

(7) Let $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Solve each of the following initial-value problems.

(a) $\mathrm{D}^2 y + 4\mathrm{D}y + 4y = 0\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,.$

(b) $\mathrm{D}^2 w + 9w = 20e^t\,, \qquad w(0) = 0\,, \quad w'(0) = 0\,.$

(8) Give a real general solution for each of the following equations.

(a) $\ddot{u} + 4\dot{u} + 5u = 3\cos(2t)$

(b) $x'' - x = t\,e^t$

(c) $y'' - y = \dfrac{1}{1+e^t}$

(9) What answer will be produced by the following MATLAB commands?

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>> syms y(t)
>> ode = diff(y,t,2) + 2*diff(y,t) + 5*y == 16*exp(t);
>> ySol(t) = dsolve(ode)
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(10) Let $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Consider the equation

$$\mathrm{L}r = \mathrm{D}^2 r - 6\mathrm{D}r + 25r = e^{t^2}\,.$$

(a) Compute the Green function $g(t)$ associated with L.

(b) Use $g(t)$ to express a particular solution $R_P(t)$ in terms of definite integrals.

(11) The functions t and t^2 are solutions of the homogeneous equation

$$t^2\frac{\mathrm{d}^2p}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}p}{\mathrm{d}t} + 2p = 0 \qquad \text{over } t > 0\,.$$

(You do not have to check that this is true!)

(a) Compute their Wronskian.

(b) Solve the initial-value problem

$$t^2\frac{\mathrm{d}^2q}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}q}{\mathrm{d}t} + 2q = t^3e^t\,, \qquad q(1) = q'(1) = 0\,, \qquad \text{over } t > 0\,.$$

Try to evaluate all definite integrals explicitly.

(12) The vertical displacement of a mass on a spring is given by

$$h(t) = 4e^{-t}\cos(7t) - 3e^{-t}\sin(7t)\,,$$

where positive displacements are upward.

(a) Express $h(t)$ in the form $h(t) = Ae^{-t}\cos(\nu t - \phi)$ with $A > 0$ and $0 \le \phi < 2\pi$, identifying the damping rate, damped period, and phase of the oscillation. (The phase may be expressed in terms of an inverse trig function.)

(b) Express $h(t)$ in the phasor form $h(t) = \mathrm{Re}\big(\gamma\, e^{\zeta t}\big)$ where γ and ζ are complex.

(c) Sketch the solution over $0 \le t \le 2$.

(13) When a 4 gram mass is hung vertically from a spring, at rest it stretches the spring 9.8 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) At $t = 0$ the mass is displaced 3 cm above its rest position and released with no initial velocity. A dashpot imparts a damping force of 2 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. There are no other forces. (Assume that the spring force is proportional to displacement and that the damping force is proportional to velocity.)

(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

(b) Find the natural frequency of the spring.

(c) Show that the system is under damped.

(d) Find the damping rate and damped frequency of the system.