Solutions of the Sample Problems for the First In-Class Exam
Math 246, Fall 2019, Professor David Levermore

(1) (a) Sketch the graph that would be produced by the following Matlab command.

fplot(@(t) 2/t, [1,6])

Solution. Your sketch should show a decreasing, concave up function that decreases from a value of 2 to a value of $\frac{1}{3}$ over the interval $[1,6]$.

(b) Sketch the graph that would be produced by the following Matlab commands.

[X, Y] = meshgrid(−5:0.1:5,−5:0.1:5)
contour(X, Y, X.ˆ2 + Y.ˆ2, [1, 9, 25])
axis square

Solution. Your sketch should show both x and y axes marked from -5 to 5 and circles of radius 1, 3, and 5 centered at the origin.

(2) Find the explicit solution for each of the following initial-value problems and identify its interval of definition.

(a) $\dfrac{\mathrm{d}z}{\mathrm{d}t} = \dfrac{\cos(t) - z}{1+t}\,, \quad z(0) = 2.$

Solution. The differential equation is a *nonhomogeneous, linear* equation in z. Its linear normal form is

$$\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{z}{1+t} = \frac{\cos(t)}{1+t}\,.$$

Because the initial time is $t = 0$ while both the coefficient and the forcing are undefined at $t = -1$ and are continuous elsewhere, we see that the *interval of definition* for the solution of the initial-value problem is $t > -1$.

We read off from its normal form that an *integrating factor* for the differential equation is given by

$$\exp\left(\int_0^t \frac{1}{1+s}\,\mathrm{d}s\right) = \exp\big(\log(1+t)\big) = 1 + t\,,$$

Upon multiplying the normal form of the equation by $(1 + t)$, we obtain the *integrating factor form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((1+t)z\big) = \cos(t)\,.$$

This is then integrated to obtain

$$(1+t)z = \sin(t) + c\,.$$

The integration constant c is found through the initial condition $z(0) = 2$ by setting $t = 0$ and $z = 0$, whereby

$$c = (1+0)2 - \sin(0) = 2\,.$$

Hence, upon solving explicitly for z, the solution is

$$z = \frac{2 + \sin(t)}{1 + t} .$$

Because this function is undefined at $t = -1$ while the initial time is $t = 0$, the interval of definition for this solution is $t > -1$, which is what we found earlier.

(b) $\dfrac{\mathrm{d}u}{\mathrm{d}z} = e^u + 1 , \quad u(0) = 0.$

Solution. The differential equation is *autonomus* (and therefore is *separable*). Its right-hand side is continuous everywhere. It is defined everywhere. It has no stationary solutions. Its separated differential form is

$$\frac{1}{e^u + 1}\, \mathrm{d}u = \mathrm{d}z .$$

This equation can be integrated to obtain

$$z = \int \frac{1}{e^u + 1}\, \mathrm{d}u = \int \frac{e^{-u}}{1 + e^{-u}}\, \mathrm{d}u = -\log(1 + e^{-u}) + c .$$

The integration constant c is found through the initial condition $u(0) = 0$ by setting $z = 0$ and $u = 0$, whereby

$$c = 0 + \log(1 + e^0) = \log(2) .$$

Hence, the solution is given implicitly by

$$z = -\log(1 + e^{-u}) + \log(2) = -\log\left(\frac{1 + e^{-u}}{2}\right) .$$

This may be solved for u as follows:

$$\begin{aligned} e^{-z} &= \frac{1 + e^{-u}}{2} , \\ 2e^{-z} - 1 &= e^{-u} , \\ u &= -\log(2e^{-z} - 1) . \end{aligned}$$

Here we need $2e^{-z} - 1 > 0$ for the log to be defined. It follows that

$$\begin{aligned} e^{-z} &> \tfrac{1}{2} , \\ -z &> \log(\tfrac{1}{2}) = -\log(2) . \end{aligned}$$

Therefore the interval of definition for this solution is $z < \log(2)$.

(c) $\dfrac{\mathrm{d}v}{\mathrm{d}t} = -3t^2 e^{-v} , \quad v(2) = 0.$

Solution. The differential equation is *separable*, but is not autonomous. Its right-hand side is continuous everywhere. It has no stationary solutions. Its separated differential form is

$$e^v\, \mathrm{d}v = -3t^2\, \mathrm{d}t .$$

This can be integrated to obtain

$$e^v = -t^3 + c .$$

The initial condition $v(2) = 0$ implies that $c = e^0 + 2^3 = 1 + 8 = 9$. Therefore $e^v = -t^3 + 9$, which can be solved as

$$v = \log(9 - t^3)\,, \quad \text{with interval of definition } t < 9^{\frac{1}{3}}\,.$$

Here we need $9 > t^3$ for the log to be defined. By taking the cube root of both sides of this inequality we obtain $9^{\frac{1}{3}} > t$. Therefore the interval of definition for this solution is $t < 9^{\frac{1}{3}}$.

(3) Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{t^2 - 4}\,x = \frac{1}{\sin(t)}\,, \qquad x(1) = 0\,.$$

(You do not need to solve this equation to answer this question, but reasoning must be given!)

Solution. The differential equation is a *nonhomogeneous, linear* equation in x and is already in normal form. The coefficient $1/(t^2 - 4)$ is undefined at $t = \pm 2$ and is continuous elsewhere. The forcing $1/\sin(t)$ is undefined where $t = n\pi$ for some integer n and is continuous elsewhere. (Alternatively, it is continuous everywhere except $t = 0,\ \pm\pi,\ \pm 2\pi,\ \cdots$, at which it is undefined.) The initial time is $t = 1$. Therefore we can see that the interval of definition for the solution of the initial-value problem is $(0, 2)$ because:

- the initial time $t = 1$ is in $(0, 2)$,
- both the coefficient and forcing are continuous over $(0, 2)$,
- the forcing is undefined at $t = 0$,
- the coefficient is undefined at $t = 2$.

(4) Consider the following Matlab commands.

```
>> [T, Y] = meshgrid(−5.0:1.0:5.0,−5.0:1.0:5.0);
>> S = T.ˆ2 − Y.ˆ3;
>> L = sqrt(1 + S.ˆ2);
>> quiver(T, Y, 1./L, S./L, 0.5)
>> axis tight, xlabel ’t’, ylabel ’y’
```

(a) What is the differential equation being studied?
(b) What kind of graph will these Matlab commands produce?

Solution (a). The differential equation being studied is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 - y^3\,.$$

This can be read off from the second command.

Solution (b). These Matlab commands will produce a direction field for the above equation in the rectangle $[-5, 5] \times [-5, 5]$ with an 11×11 grid of arrows. The fact it is producing a direction field is seen from the third and fourth commands. The sizes of the rectangle and the grid can be read off from the first command.

(5) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y^2(y+2)(y-4)}{y-2}\,.$$

(a) Sketch its phase-line portrait over the interval $[-6, 6]$. Identify points where it is undefined. Identify its stationary points and classify each as being either stable, unstable, or semistable.

Solution. This equation is autonomous. Its right-hand side is undefined at $y = 2$ and is differentiable elsewhere. Its stationary points are found by setting

$$\frac{y^2(y+2)(y-4)}{y-2} = 0\,.$$

Therefore the stationary points are $y = -2$, $y = 0$, and $y = 4$. Because the right-hand side is differentiable at each of these stationary points, no other solutions will will touch them. (Uniqueness!)

A sign analysis of the right-hand side shows that the phase-line portrait is

$$\begin{array}{ccccccccccc}
 & - & & + & & + & & - & & + & \\
\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow & & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \circ & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & y \\
-6 & & -2 & & 0 & & 2 & & 4 & & 6 \\
 & & \text{unstable} & & \text{semistable} & & \text{undefined} & & \text{unstable} & &
\end{array}$$

Remark. The terms stable, unstable, and semistable are applied to solutions. The point $y = 2$ is not a solution, so these terms should not be applied to it.

(b) For each stationary point identify the set of initial values $y(0)$ such that the solution $y(t)$ converges to that stationary point as $t \to \infty$.

Solution. As t increases the solutions will move in the direction of the arrows shown in the phase-line portrait given in the solution to part (a). By uniqueness any nonstationary solution will not touch a stationary one.

- Because the stationary point -2 is unstable $y(t) \to -2$ as $t \to \infty$ if and only if $y(0) = -2$.
- Because the stationary point 0 is semistable $y(t) \to 0$ as $t \to \infty$ if and only if $y(0)$ is in $(-2, 0]$.
- Because the stationary point 4 is unstable $y(t) \to 4$ as $t \to \infty$ if and only if $y(0) = 4$.

(c) For each stationary point identify the set of initial values $y(0)$ such that the solution $y(t)$ converges to that stationary point as $t \to -\infty$.

Solution. As t decreases the solutions will move in the opposite direction of the arrows shown in the phase-line portrait given in the solution to part (a). By uniqueness any nonstationary solution will not touch a stationary one.

- Because the stationary point -2 is unstable $y(t) \to -2$ as $t \to -\infty$ if and only if $y(0)$ is in $(-\infty, 0)$.
- Because the stationary point 0 is semistable $y(t) \to 0$ as $t \to -\infty$ if and only if $y(0)$ is in $[0, 2)$.
- Because the stationary point 4 is unstable $y(t) \to 4$ as $t \to -\infty$ if and only if $y(0)$ is in $(2, \infty)$.

(d) Identify all initial values $y(0)$ such that the interval of definition of the solution $y(t)$ is $(-\infty, \infty)$.

Solution. The interval of definition the solution $y(t)$ is $(-\infty, \infty)$ when either $y(0)$ is in $[-2, 0]$ or $y(0) = 4$.

Remark. When $y(0)$ is in either $(-\infty, -2)$ or $(4, \infty)$ then when $|y(t)|$ gets large $y(t)$ will behave like a nonzero solution of

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^3 \,,$$

all of which blow up in finite time as t increases.

Remark. When $y(0)$ is in either $(0, 2)$ or $(2, 4)$ then as $y(t)$ approaches 2 it will behave like a solution of

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{32}{y-2} \,,$$

for all of which $y'(t)$ blows up in finite time as t increases.

(e) Sketch a graph of y versus t showing several solution curves. The graph should show all of the stationary solutions as well as solution curves above and below each of them. Every value of y for which the equation is defined should lie on at least one sketched solution curve.

Solution. This will be given during the review session. You can sketch it below.

(6) In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population such that it would triple every five weeks. There are 85,000 mosquitoes in the area when a flock of birds arrives that eats 25,000 mosquitoes per week. Write down an initial-value problem that governs $M(t)$, the population of mosquitoes in the area after the flock of birds arrives. (You do not have to solve the initial-value problem!)

Solution. The population tripling every five weeks corresponds to a growth factor of $3^{\frac{t}{5}} = \left(e^{\log(3)}\right)^{\frac{t}{5}} = e^{\frac{1}{5}\log(3)t}$, which implies a growth rate of $\frac{1}{5}\log(3)$. Therefore the initial-value problem that M satisfies is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \tfrac{1}{5}\log(3)M - 25,000\,, \qquad M(0) = 85,000\,.$$

Remark. This equation is a nonhomogeneous linear equation that is also autonomous. If you were asked for an analytic solution then it is best to think of it as linear. However, for some questions it is better to think of it as autonomous. For example, if you are asked whether the flock of birds is big enough to control the mosquitoes then a phase-line portrait is a quick route to the answer.

(7) A tank initially contains 100 liters of pure water. Beginning at time $t = 0$ brine (salt water) with a salt concentration of 2 grams per liter (gr/lit) flows into the tank at a constant rate of 3 liters per minute (lit/min) and the well-stirred mixture flows out of the tank at the same rate. Let $S(t)$ denote the mass (gr) of salt in the tank at time $t \geq 0$.

(a) Write down an initial-value problem that governs $S(t)$.

Solution. Because brine flows in and out of the tank at the same rate, the tank will contain 100 liters of brine for every $t > 0$. The salt concentration of the brine in the tank at time t will therefore be $S(t)/100$ gr/lit. Because this is also the concentration of the outflow, $S(t)$, the mass of salt in the tank at time t, will satisfy

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \text{RATE IN} - \text{RATE OUT} = 2 \cdot 3 - \frac{S}{100} \cdot 3 = 6 - \tfrac{3}{100}S\,.$$

Because there is no salt in the tank initially, the initial-value problem that governs $S(t)$ is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 6 - \tfrac{3}{100}S\,, \qquad S(0) = 0\,.$$

(b) Is $S(t)$ an increasing or decreasing function of t? (Give your reasoning.)

Solution. We see from part (a) that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \tfrac{3}{100}(200 - S) > 0 \quad \text{for } S < 200\,,$$

whereby $S(t)$ is an increasing function of t that will approach the stationary value of 200 gr as $t \to \infty$.

(c) What is the behavior of $S(t)$ as $t \to \infty$? (Give your reasoning.)

Solution. The argument given for part (b) already shows that $S(t)$ is an increasing function of t that approaches the stationary value of 200 gr as $t \to \infty$.

(d) Derive an explicit formula for $S(t)$.

Solution. The differential equation given in the answer to part (a) is linear, so write it in the form

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \tfrac{3}{100} S = 6\,.$$

An integrating factor is $e^{\frac{3}{100}t}$, whereby

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{\frac{3}{100}t} S\big) = 6 e^{\frac{3}{100}t}\,.$$

This is the integrated to obtain

$$e^{\frac{3}{100}t} S = 200 e^{\frac{3}{100}t} + c\,.$$

The integration constant c is found by setting $t = 0$ and $S = 0$, whereby

$$c = e^0 \cdot 0 - 200 \cdot e^0 = -200\,.$$

Then solving for S gives

$$S(t) = 200 - 200 e^{-\frac{3}{100}t}\,.$$

(e) How does the answer to part (a) change if the well-stirred mixture flows out of the tank at a constant rate of 2 liters per minute?

Solution. Because brine flows into the tank at 3 lit/min and out of it at 2 lit/min, the tank will contain $100 + t$ liters of brine for every $t > 0$. The salt concentration of the brine in the tank at time t will therefore be $S(t)/(100 + t)$ gr/lit. Because this is also the concentration of the outflow, $S(t)$, the mass of salt in the tank at time t, will satisfy

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \text{RATE IN} - \text{RATE OUT} = 2 \cdot 3 - \frac{S}{100 + t} \cdot 2 = 6 - \frac{2}{100 + t}\, S\,.$$

Because there is no salt in the tank initially, the initial-value problem that governs $S(t)$ is

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 6 - \frac{2}{100 + t}\, S\,, \qquad S(0) = 0\,.$$

Remark. Can you see how the answers to parts (b-d) change for this problem?

(8) A 2 kilogram (kg) mass initially at rest is dropped in a medium that offers a resistance of $v^2/40$ newtons (= kg m/sec^2) where v is the downward velocity (m/sec) of the mass. The gravitational acceleration is 9.8 m/sec^2.

(a) What is the terminal velocity of the mass?

Solution. The terminal velocity is the velocity at which the force of resistence balances that of gravity. This happens when

$$\tfrac{1}{40}v^2 = mg = 2 \cdot 9.8\,.$$

Upon solving this for v we obtain

$$\begin{aligned} v &= \sqrt{40 \cdot 2 \cdot 9.8}\ \text{m/sec} \qquad\qquad \text{(full marks)} \\ &= \sqrt{4 \cdot 2 \cdot 98} = \sqrt{4 \cdot 2 \cdot 2 \cdot 49} = \sqrt{4^2 \cdot 7^2} = 4 \cdot 7 = 28\ \text{m/sec}\,. \end{aligned}$$

(b) Write down an initial-value problem that governs v as a function of time. (You do not have to solve it!)

Solution. The net downward force on the falling mass is the force of gravity minus the force of resistence. By Newton ($ma = F$), this leads to

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - \tfrac{1}{40}v^2\,.$$

Because $m = 2$ and $g = 9.8$, and because the mass is initially at rest, this yields the initial-value problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 9.8 - \tfrac{1}{80}v^2\,, \qquad v(0) = 0\,.$$

Remark. You should be able to solve this initial-value problem.

(9) Give an implicit general solution to each of the following differential equations.

(a) $\displaystyle\left(\frac{y}{x} + 3x\right)\mathrm{d}x + \big(\log(x) - y\big)\,\mathrm{d}y = 0\,.$

Solution. This equation is neither linear nor separable. Because

$$\partial_y\left(\frac{y}{x} + 3x\right) = \frac{1}{x} \qquad = \qquad \partial_x\big(\log(x) - y\big) = \frac{1}{x}\,,$$

the equation is *exact*. Therefore we can find $H(x, y)$ such that

$$\partial_x H(x, y) = \frac{y}{x} + 3x\,, \qquad \partial_y H(x, y) = \log(x) - y\,.$$

The first of these equations implies that

$$H(x, y) = y\log(x) + \tfrac{3}{2}x^2 + h(y)\,.$$

Plugging this into the second equation then shows that

$$\log(x) - y = \partial_y H(x, y) = \log(x) + h'(y)\,.$$

Hence, $h'(y) = -y$, which yields $h(y) = -\frac{1}{2}y^2$. Therefore a general solution is governed implicitly by

$$y\log(x) + \tfrac{3}{2}x^2 - \tfrac{1}{2}y^2 = c\,, \quad \text{where } c \text{ is an arbitrary constant}\,.$$

(b) $(x^2 + y^3 + 2x)\,\mathrm{d}x + 3y^2\,\mathrm{d}y = 0\,.$

Solution. This equation is neither linear nor separable. Because

$$\partial_y(x^2 + y^3 + 2x) = 3y^2 \qquad \neq \qquad \partial_x(3y^2) = 0\,,$$

the equation is *not exact*. Seek an integrating factor $\rho(x, y)$ such that

$$\partial_y\big((x^2 + y^3 + 2x)\rho\big) = \partial_x(3y^2\rho)\,.$$

This means that ρ must satisfy

$$(x^2 + y^3 + 2x)\partial_y\rho + 3y^2\rho = 3y^2\partial_x\rho\,.$$

If we assume that ρ depends only on x (so that $\partial_y\rho = 0$) then this reduces to

$$\rho = \partial_x\rho\,,$$

which depends only on x. We see from this that $\rho = e^x$ is an integrating factor. This implies that

$$(x^2 + y^3 + 2x)e^x\,\mathrm{d}x + 3y^2e^x\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

Therefore we can find $H(x, y)$ such that

$$\partial_x H(x, y) = (x^2 + y^3 + 2x)e^x\,, \qquad \partial_y H(x, y) = 3y^2e^x\,.$$

The second of these equations implies that

$$H(x, y) = y^3e^x + h(x)\,.$$

Plugging this into the first equation then yields

$$(x^2 + y^3 + 2x)e^x = \partial_x H(x, y) = y^3e^x + h'(x)\,.$$

Hence, h satisfies

$$h'(x) = (x^2 + 2x)e^x\,.$$

This can be integrated to obtain $h(x) = x^2e^x$. Therefore a general solution is governed implicitly by

$$(y^3 + x^2)e^x = c\,, \quad \text{where } c \text{ is an arbitrary constant}\,.$$

(10) Suppose we are using the Runge-midpoint method to numerically approximate the solution of an initial-value problem over the time interval $[0, 5]$. By what factor would we expect the error to decrease when we increase the number of time steps taken from 500 to 2000?

Solution. The Runge-midpoint method is second order, which means its (global) error scales like h^2 where h is the step size. When the number of time steps taken increases from 500 to 2000, the step size h decreases by a factor of $1/4$. Therefore the error will decrease (like h^2) by a factor of $1/4^2 = 1/16$.

Remark. You should be able to answer similar questions about the explicit Euler, Runge-trapezoidal, and Runge-Kutta methods.

(11) Consider the following Matlab function m-file.

```
function [t,y] = solveit(tI, yI, tF, n)

t = zeros(n + 1, 1); y = zeros(n + 1, 1);
t(1) = tI; y(1) = yI; h = (tF - tI)/n;
for i = 1:n
z = t(i)^4 + y(i)^2;
t(i + 1) = t(i) + h;
y(i + 1) = y(i) + (h/2)*(z + t(i + 1)^4 + (y(i) + h*z)^2);
end
```

Suppose the input values are tI = 1, yI = 1, tF = 5, and n = 20.

(a) What is the initial-value problem being approximated numerically?

Solution. The initial-value problem being approximated is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t^4 + y^2 \,, \qquad y(1) = 1 \,.$$

Remark. You should not confuse the $y(1)$ above with the y(1) appearing in the Matlab program. The $y(1)$ denotes the solution $y(t)$ of the initial-value problem evaluated at $t = 1$. The y(1) denotes the first entry of the Matlab array y. Here they have the same value because tI = 1 but they will be different in general.

(b) What is the numerical method being used?

Solution. The Runge-Trapezoidal method is being used.

(c) What is the step size?

Solution. Because tF = 5, tI = 1, and n = 20, the step size is

$$\mathrm{h} = \frac{\mathrm{tF} - \mathrm{tI}}{\mathrm{n}} = \frac{5-1}{20} = \frac{4}{20} = .2 \,.$$

Remark. You must plug in the correct values for tF, tI, and n to get any credit.

(d) What are the output values of t(2) and y(2)?

Remark. Notice that this is asking for the values of the second entries of the Matlab arrays t and y produced by the above m-file. In particular, y(2) is not the solution $y(t)$ of the initial-value problem evaluated at $t = 2$!

Solution. The step size is given by h = .2. The initial time and value are given by t(1) = tI = 1 and y(1) = yI = 1. By setting i = 1 inside the "for loop" we see that

$$\begin{aligned}
\mathrm{z} &= \mathrm{t}(1)^4 + \mathrm{y}(1)^2 = 1 + 1 = 2 \,, \\
\mathrm{t}(2) &= \mathrm{t}(1) + \mathrm{h} = 1 + .2 = 1.2 \,, \\
\mathrm{y}(2) &= \mathrm{y}(1) + (\mathrm{h}/2)\,\big(\mathrm{z} + \mathrm{t}(2)^4 + (\mathrm{y}(1) + \mathrm{h}\ \mathrm{z})^2\big) \\
&= 1 + .1\big(2 + (1.2)^4 + (1 + .2 \cdot 2)^2\big) \,.
\end{aligned}$$

(12) Suppose we have used a numerical method to approximate the solution of an initial-value problem over the time interval $[1, 6]$ with 1000 uniform time steps. How many uniform time steps do we need to reduce the global error of our approximation by roughly a factor of $\frac{1}{81}$ if the method we had used was each of the following?

(a) Explicit Euler method
(b) Runge-trapezoidal method
(c) Runge-midpoint method
(d) Runge-Kutta method

Solution (a). The explicit Euler method is first order, so its error scales like h. To reduce the error by a factor of $\frac{1}{81}$, we must reduce h by a factor of $\frac{1}{81}$. We must increase the number of time steps by a factor of 81, which means we need $81,000$ uniform time steps.

Solution (b). The Runge-trapezoidal method is second order, so its error scales like h^2. To reduce the error by a factor of $\frac{1}{81}$, we must reduce h by a factor of $\frac{1}{81}^{\frac{1}{2}} = \frac{1}{9}$. We must increase the number of time steps by a factor of 9, which means you need $9,000$ uniform time steps.

Solution (c). The Runge-midpoint method is second order, so its error scales like h^2. To reduce the error by a factor of $\frac{1}{81}$, we must reduce h by a factor of $\frac{1}{81}^{\frac{1}{2}} = \frac{1}{9}$. We must increase the number of time steps by a factor of 9, which means you need $9,000$ uniform time steps.

Solution (d). The Runge-Kutta method is fourth order, so its error scales like h^4. To reduce the error by a factor of $\frac{1}{81}$, we must reduce h by a factor of $\frac{1}{81}^{\frac{1}{4}} = \frac{1}{3}$. We must increase the number of time steps by a factor of 3, which means you need $3,000$ uniform time steps.