Final Exam, Math 246/246H
Wednesday, 11 December 2019

Closed book. No electronics. Answer only one question on each answer sheet. Write your name and which question is being answered on each answer sheet. Sign the Honor Pledge on the first answer sheet only. Indicate your answer to each part of each question clearly. Indicate on the front of an answer sheet if your work continues on the back. Cross out work that you do not want considered. Give reasoning that justifies your answers.

(1) [22] Find an explicit solution to each of the following initial-value problems. Identify their intervals of definition.

(a) [11] $(1+t)\,w' + 2\,w = \dfrac{3\cos(t)}{1+t}\,, \qquad w(0) = 5\,.$

(b) [11] $u' = \dfrac{t}{u\sqrt{t^2-4}}\,, \qquad u(-5) = -2\,.$

(2) [10] Consider the following Matlab commands.

```
[t,y] = ode45(@(t,y) y.*(y−6), [2,7], −3.0:6.0:9.0);
plot(t,y)
```

The following questions need not be answered in Matlab format!

(a) [2] What differential equation is being solved numerically?
(b) [3] Give the initial condition for each solution being approximated.
(c) [1] Over what time interval are the solutions being approximated?
(d) [4] Sketch each of these solutions over this time interval on a single graph. Label the initial value of each solution clearly.

(3) [6] Suppose the Runge-trapezoidal (improved Euler) method is used to approximately solve an initial-value problem over the interval $[2, 10]$ with 500 uniform time steps.

(a) [3] How many uniform time steps are needed to reduce the global error by a factor of $\frac{1}{64}$?
(b) [3] What is the order of a numerical method that reduces the global error by a factor of $\frac{1}{64}$ when the step size is halved?

(4) [22] Give an explicit real-valued general solution to each of the following equations.

(a) [11] $u'' + 8u' + 16u = 16\cos(4t)\,.$
(b) [11] $v'''' + 3v'' - 4v = 10e^t\,.$

(5) [10] A spring stretches 9.8 cm when a 10 gram mass is hung from it. (Gravitational acceleration is $g = 980$ cm/sec^2.) A damper imparts a damping force of 1000 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 4 cm/sec. Assume that the spring force is proportional to displacement, that the damping force is proportional to velocity, and that there are no other forces. At $t = 0$ the mass is displaced 2 cm below its rest position and is released with a upward velocity of 3 cm/sec.

(a) [7] Give an initial-value problem that governs the displacement $h(t)$ for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)
(b) [3] Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

More Problems on the Other Side!

(6) [10] Given that t and $t\log(t)$ solve the associated homogeneous differential equation, solve (evaluating any definite integrals that arise) the initial-value problem

$$t^2v'' - t\,v' + v = t\,\big(\log(t)\big)^2\,, \qquad v(1) = 0\,, \quad v'(1) = 0\,.$$

(7) [22] Let $x(t)$ be the solution of the initial-value problem

$$x'' - 8x' + 25x = f(t)\,, \qquad x(0) = 2\,, \quad x'(0) = -3\,,$$

where $f(t) = t^2 + u(t-1)\big(e^{1-t} - t^2\big)$. Here u is the unit step function.

(a) [11] Find the Laplace transform $F(s)$ of the forcing $f(t)$.

(b) [11] Find the Laplace transform $X(s)$ of the solution $x(t)$.
(DO NOT take the inverse Laplace transform to find $x(t)$; just solve for $X(s)$!)

You may refer to the table.

(8) [10] Find the function $y(t)$ whose Laplace transform is $Y(s) = \dfrac{e^{-2s}(s+8)}{s^2 - 2s + 10}$.
You may refer to the table.

(9) [22] A real 2×2 matrix $\mathbf{C}$ has the eigenpairs

$$\left(-2\,, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) \quad \text{and} \quad \left(-3\,, \begin{pmatrix} 3 \\ -1 \end{pmatrix}\right)\,.$$

(a) [4] Give a general solution to the linear planar system $\mathbf{x}' = \mathbf{C}\mathbf{x}$.

(b) [4] Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ that diagonalize $\mathbf{C}$.

(c) [8] Compute $e^{t\mathbf{C}}$.

(d) [6] Sketch a phase-plane portrait for this system and identify its type. Classify the stability of the origin. (Carefully mark all sketched orbits with arrows!)

(10) [22] Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}^{\mathrm{I}}$ for the following $\mathbf{A}$ and $\mathbf{x}^{\mathrm{I}}$.

(a) [11] $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$, $\qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

(b) [11] $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}$, $\qquad \mathbf{x}^{\mathrm{I}} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(11) [22] Consider the system

$$x' = 12 + 2y - 3x^2\,, \qquad y' = 6xy\,.$$

(a) [11] Find a nonconstant function $H(x, y)$ such that every orbit of the system satisfies $H(x, y) = c$ for some constant c.

(b) [11] The stationary points are $(-2, 0)$, $(2, 0)$, and $(0, -6)$. In the phase-plane sketch these stationary points plus the level set $H(x, y) = c$ for each value of c that corresponds to a stationary point that is a saddle. (No arrows are required!)

(12) [22] Consider the system

$$\dot{u} = 3u - v\,, \qquad \dot{v} = 5u - 3v + 2u^2\,.$$

(a) [4] This system has two stationary points. Find them.

(b) [4] Compute the Jacobian matrix at each stationary point.

(c) [8] Classify the type and stability of each stationary point.

(d) [6] Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. (Carefully mark all sketched orbits with arrows!)