Third In-Class Exam Solutions
Math 246, Professor David Levermore
Thursday, 14 November 2019

(1) [6] Recast the ordinary differential equation $y'''' - e^y y''' + e^t y'' - \sin(t + y') = 0$ as a first-order system of ordinary differential equations.

Solution. The normal form of the equation is

$$y'''' = e^y y''' - e^t y'' + \sin(t + y') \,.$$

Because this equation is fourth order, the first-order system must have dimension at least four. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ e^{x_1} x_4 - e^t x_3 + \sin(t + x_2) \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix}.$$

Remark. There should be no y, y', y'', or y''' appearing in the first-order system. The only place these should appear is in the dictionary on the right that shows their relationship to the new variables. The first-order system should be expressed solely in terms of the new variables, which are x_1, x_2, x_3, and x_4 in the solution given above. Any letter except y could have been used for the new variables.

(2) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^2 \\ -1 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

(a) [2] Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{C}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{C}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system found in part (b).

(d) [3] Compute the Green matrix associated with the system found in part (b).

Solution (a). The Wronskian is

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^2 & e^t \\ -1 & e^t \end{pmatrix} = t^2 \cdot e^t - (-1) \cdot e^t = (t^2 + 1)e^t \,.$$

Solution (b). If $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions for the system $\mathbf{x}' = \mathbf{C}(t)\mathbf{x}$ then a fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{pmatrix} t^2 & e^t \\ -1 & e^t \end{pmatrix}.$$

Because any fundamental matrix is invertible and satisfies $\mathbf{\Psi}'(t) = \mathbf{C}(t)\mathbf{\Psi}(t)$, we see that

$$\begin{aligned} \mathbf{C}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}(t)^{-1} &= \begin{pmatrix} 2t & e^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} t^2 & e^t \\ -1 & e^t \end{pmatrix}^{-1} \\ &= \frac{1}{(t^2+1)e^t} \begin{pmatrix} 2t & e^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} e^t & -e^t \\ 1 & t^2 \end{pmatrix} \\ &= \frac{1}{(t^2+1)e^t} \begin{pmatrix} 2te^t + e^t & -2te^t + t^2 e^t \\ e^t & t^2 e^t \end{pmatrix}. \end{aligned}$$

Remark. The solution can be simplified to

$$\mathbf{C}(t) = \frac{1}{t^2+1}\begin{pmatrix} 2t+1 & -2t+t^2 \\ 1 & t^2 \end{pmatrix},$$

but this simplification was not required for full credit.

Solution (c). A general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} t^2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} e^t \\ e^t \end{pmatrix}.$$

Solution (d). By using the fundamental matrix $\mathbf{\Psi}(t)$ from part (b) we find that the Green matrix is

$$\begin{aligned}
\mathbf{G}(t,s) = \mathbf{\Psi}(t)\mathbf{\Psi}(s)^{-1} &= \begin{pmatrix} t^2 & e^t \\ -1 & e^t \end{pmatrix}\begin{pmatrix} s^2 & e^s \\ -1 & e^s \end{pmatrix}^{-1} \\
&= \frac{1}{(s^2+1)e^s}\begin{pmatrix} t^2 & e^t \\ -1 & e^t \end{pmatrix}\begin{pmatrix} e^s & -e^s \\ 1 & s^2 \end{pmatrix} \\
&= \frac{1}{(s^2+1)e^s}\begin{pmatrix} t^2e^s + e^t & -t^2e^s + s^2e^t \\ -e^s + e^t & e^s + s^2e^t \end{pmatrix}.
\end{aligned}$$

Notice that $\mathbf{G}(s,s) = \mathbf{I}$.

(3) [6] Given that 2 is an eigenvalue of the matrix

$$\mathbf{C} = \begin{pmatrix} 4 & 0 & -4 \\ 0 & 3 & 3 \\ 2 & 2 & 4 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{C}$ associated with 2.

Solution. The eigenvectors of $\mathbf{C}$ associated with 2 are all nonzero vectors $\mathbf{v}$ such that $\mathbf{C}\mathbf{v} = 2\mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ such that $(\mathbf{C} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$, which is

$$\begin{pmatrix} 2 & 0 & -4 \\ 0 & 1 & 3 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned}
2v_1 \qquad\quad - 4v_3 &= 0\,. \\
v_2 + 3v_3 &= 0\,, \\
2v_1 + 2v_2 + 2v_3 &= 0\,,
\end{aligned}$$

This system may be solved either by elimination or by row reduction. By any method its general solution is found to be

$$v_1 = 2\alpha\,, \quad v_2 = -3\alpha\,, \quad v_3 = \alpha\,, \qquad \text{for any constant } \alpha\,.$$

Therefore every eigenvector of $\mathbf{C}$ associated with 2 has the form

$$\alpha\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \qquad \text{for some constant } \alpha \neq 0\,.$$

(4) [10] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -5 & -4 \\ 1 & -1 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 6z + 9 = (z+3)^2 .$$

This is a perfect square with $\mu = -3$. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-3t}\left[\mathbf{I} + t\left(\mathbf{A} - (-3)\mathbf{I}\right)\right] \\ &= e^{-3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix}\right] = e^{-3t}\begin{pmatrix} 1-2t & -4t \\ t & 1+2t \end{pmatrix}. \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A} + 3\mathbf{I}) = 0$!) Therefore the solution of the initial-value problem is

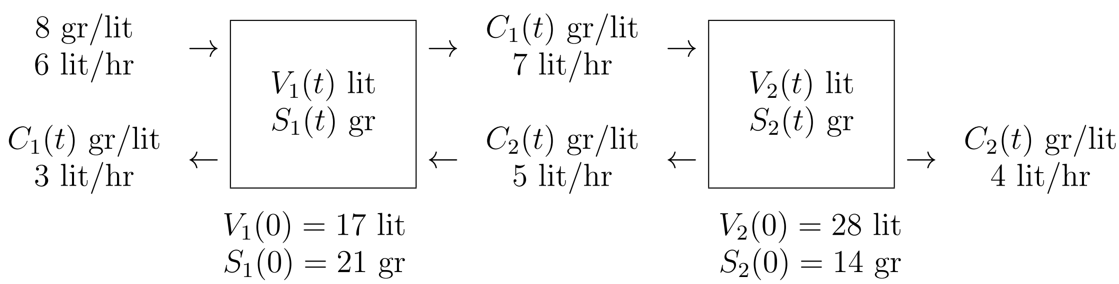
$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}^I = e^{-3t}\begin{pmatrix} 1-2t & -4t \\ t & 1+2t \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = e^{-3t}\begin{pmatrix} -8t \\ 2+4t \end{pmatrix}.$$

(5) [8] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 17 liters and the second contains 28 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 21 grams of salt in the first tank and 14 grams in the second.

(a) [6] Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(b) [2] Give the interval of definition for the solution of this initial-value problem.

Solution (a). Let $V_1(t)$ and $V_2(t)$ be the volumes (lit) of brine in the first and second tank at time t hours. Let $S_1(t)$ and $S_2(t)$ be the mass (gr) of salt in the first and second tank at time t hours. Because the mixtures are assumed to be well-stirred, the salt concentration of the brine in the tanks at time t are $C_1(t) = S_1(t)/V_1(t)$ and $C_2(t) = S_2(t)/V_2(t)$ respectively. In particular, these are the salt concentrations of the brine that flows out of these tanks. We have the following picture.



We are asked to write down an initial-value problem that governs $S_1(t)$ and $S_2(t)$.

The rates work out so there will be $V_1(t) = 17 + t$ liters of brine in the first tank and $V_2(t) = 28 - 2t$ liters in the second. Then $S_1(t)$ and $S_2(t)$ are governed by the initial-value problem

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = 8 \cdot 6 + \frac{S_2}{28 - 2t} 5 - \frac{S_1}{17 + t} 7 - \frac{S_1}{17 + t} 3 \,, \qquad S_1(0) = 21 \,,$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{S_1}{17 + t} 7 - \frac{S_2}{28 - 2t} 5 - \frac{S_2}{28 - 2t} 4 \,, \qquad S_2(0) = 14 \,.$$

Your answer could be left in the above form. However, it can be simplified to

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = 48 + \frac{5}{28 - 2t} S_2 - \frac{10}{17 + t} S_1 \,, \qquad S_1(0) = 21 \,,$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{7}{17 + t} S_1 - \frac{9}{28 - 2t} S_2 \,, \qquad S_2(0) = 14 \,.$$

Solution (b). This first-order system of differential equations is *linear*. Its coefficients are undefined either at $t = 14$ or $t = -17$ and are continuous elsewhere. Its forcing is constant, so is continuous everywhere. Therefore the natural interval of definition for the solution of this initial-value problem is $(-17, 14)$ because:

- the initial time $t = 0$ is in $(-17, 14)$;
- all the coefficients and the forcing are continuous over $(-17, 14)$;
- every coefficient of S_1 is undefined at $t = -17$;
- every coefficient of S_2 is undefined at $t = 14$.

However, it could also be argued that the interval of definition for the solution of this initial-value problem is $[0, 14)$ because the word problem starts at $t = 0$.

(6) [8] A 4×4 matrix $\mathbf{K}$ has the eigenpairs

$$\left(0 \,, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right) , \qquad \left(1 \,, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}\right) , \qquad \left(4 \,, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right) , \qquad \left(9 \,, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\right) ,$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{K}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{K}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{K}\mathbf{x}$.

Solution (a). One choice for $\mathbf{V}$ and $\mathbf{D}$ is

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} , \qquad \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix} .$$

Remark. There are 23 other choices for $\mathbf{D}$. (Can you find them all?)

Solution (b). Use the given eigenpairs to construct the real eigensolutions

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} , \quad \mathbf{x}_2(t) = e^{t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} , \quad \mathbf{x}_3(t) = e^{4t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} , \quad \mathbf{x}_4(t) = e^{9t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} .$$

Then a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t) & \mathbf{x}_4(t) \end{pmatrix} = \begin{pmatrix} 1 & e^t & e^{4t} & e^{9t} \\ 1 & e^t & -e^{4t} & -e^{9t} \\ 1 & -e^t & e^{4t} & -e^{9t} \\ 1 & -e^t & -e^{4t} & e^{9t} \end{pmatrix}.$$

Alternative Solution (b). Given the $\mathbf{V}$ and $\mathbf{D}$ from part (a), a fundamental matrix for the system is

$$\mathbf{\Psi}(t) = \mathbf{V}e^{t\mathbf{D}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{4t} & 0 \\ 0 & 0 & 0 & e^{9t} \end{pmatrix} = \begin{pmatrix} 1 & e^t & e^{4t} & e^{9t} \\ 1 & e^t & -e^{4t} & -e^{9t} \\ 1 & -e^t & e^{4t} & -e^{9t} \\ 1 & -e^t & -e^{4t} & e^{9t} \end{pmatrix}.$$

(7) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 & -4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution by Formula. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -7 & -4 \\ 2 & -3 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 10z + 29 = (z+5)^2 + 2^2.$$

This is a sum of squares with $\mu = -5$ and $\nu = 2$. Then

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-5t}\left[\cos(2t)\mathbf{I} + \frac{\sin(2t)}{2}\left(\mathbf{A} - (-5)\mathbf{I}\right)\right] \\ &= e^{-5t}\left[\cos(2t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(2t)}{2}\begin{pmatrix} -2 & -4 \\ 2 & 2 \end{pmatrix}\right] \\ &= e^{-5t}\begin{pmatrix} \cos(2t) - \sin(2t) & -2\sin(2t) \\ \sin(2t) & \cos(2t) + \sin(2t) \end{pmatrix}. \end{aligned}$$

(Check that $\mathrm{tr}(\mathbf{A} + 5\mathbf{I}) = 0$!) Therefore a general solution is

$$\begin{aligned} \mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{c} &= e^{-5t}\begin{pmatrix} \cos(2t) - \sin(2t) & -2\sin(2t) \\ \sin(2t) & \cos(2t) + \sin(2t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{-5t}\begin{pmatrix} \cos(2t) - \sin(2t) \\ \sin(2t) \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} -2\sin(2t) \\ \cos(2t) + \sin(2t) \end{pmatrix}. \end{aligned}$$

Solution by Eigensolutions. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -7 & -4 \\ 2 & -3 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 10z + 29 = (z+5)^2 + 2^2.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $-5 + i2$ and $-5 - i2$. Consider the matrix

$$\mathbf{A} - (-5 - i2)\mathbf{I} = \begin{pmatrix} -2 + i2 & -4 \\ 2 & 2 + i2 \end{pmatrix}.$$

After checking that the determinant of this matrix is zero, we can read off from its first column that an eigenpair of $\mathbf{A}$ is

$$\left(-5 + i2\,,\,\begin{pmatrix} -1+i \\ 1 \end{pmatrix}\right).$$

(Another eigenpair is the complex conjugate of this one, but we will not need it.) This eigenpair yields the complex-valued eigensolution

$$\begin{aligned}\mathbf{x}(t) &= e^{(-5+i2)t}\begin{pmatrix} -1+i \\ 1 \end{pmatrix} = e^{-5t}\big(\cos(2t) + i\sin(2t)\big)\begin{pmatrix} -1+i \\ 1 \end{pmatrix} \\ &= e^{-5t}\begin{pmatrix} \big(\cos(2t) + i\sin(2t)\big)(-1+i) \\ \cos(2t) + i\sin(2t) \end{pmatrix} \\ &= e^{-5t}\begin{pmatrix} \big(-\cos(2t) - \sin(2t)\big) + i\big(\cos(2t) - \sin(2t)\big) \\ \cos(2t) + i\sin(2t) \end{pmatrix}.\end{aligned}$$

A fundamental set of real-valued solutions can be read off from the real and imaginary parts of this complex-valued eigensolution as

$$\mathbf{x}_1(t) = e^{-5t}\begin{pmatrix} -\cos(2t) - \sin(2t) \\ \cos(2t) \end{pmatrix}, \qquad \mathbf{x}_2(t) = e^{-5t}\begin{pmatrix} \cos(2t) - \sin(2t) \\ \sin(2t) \end{pmatrix}.$$

Therefore a real general solution is

$$\mathbf{x}(t) = c_1 e^{-5t}\begin{pmatrix} -\cos(2t) - \sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} \cos(2t) - \sin(2t) \\ \sin(2t) \end{pmatrix}.$$

(8) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution by Eigensolutions. The characteristic polynomial of $\mathbf{B} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}$ is

$$p(z) = z^2 - \mathrm{tr}(\mathbf{B})z + \det(\mathbf{B}) = z^2 - 4z - 12 = (z+2)(z-6).$$

The eigenvalues of $\mathbf{B}$ are the roots of this polynomial, which are -2 and 6. Consider the matrices

$$\mathbf{B} + 2\mathbf{I} = \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix}, \qquad \mathbf{B} - 6\mathbf{I} = \begin{pmatrix} -5 & 5 \\ 3 & -3 \end{pmatrix}.$$

After checking that the determinant of each matrix is zero, we can read off that eigenpairs of $\mathbf{B}$ are

$$\left(-2\,,\,\begin{pmatrix} 5 \\ -3 \end{pmatrix}\right), \qquad \left(6\,,\,\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right).$$

Therefore a real general solution of the system is

$$\mathbf{x}(t) = c_1 e^{-2t}\begin{pmatrix} 5 \\ -3 \end{pmatrix} + c_2 e^{6t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution by Formula. The characteristic polynomial of $\mathbf{B} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}$ is

$$p(z) = z^2 - \text{tr}(\mathbf{B})z + \det(\mathbf{B}) = z^2 - 4z - 12 = (z-2)^2 - 4 - 12 = (z-2)^2 - 4^2 \,.$$

This is a difference of squares with $\mu = 2$ and $\nu = 4$. Then

$$\begin{aligned} e^{t\mathbf{B}} &= e^{2t}\left[\cosh(4t)\mathbf{I} + \frac{\sinh(4t)}{4}\big(\mathbf{B} - 2\mathbf{I}\big)\right] \\ &= e^{2t}\left[\cosh(4t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(4t)}{4}\begin{pmatrix} -1 & 5 \\ 3 & 1 \end{pmatrix}\right] \\ &= e^{2t}\begin{pmatrix} \cosh(4t) - \frac{1}{4}\sinh(4t) & \frac{5}{4}\sinh(4t) \\ \frac{3}{4}\sinh(4t) & \cosh(4t) + \frac{1}{4}\sinh(4t) \end{pmatrix} . \end{aligned}$$

(Check that $\text{tr}(\mathbf{B} - 2\mathbf{I}) = 0$!) Therefore a real general solution of the system is

$$\begin{aligned} \mathbf{x}(t) = e^{t\mathbf{B}}\mathbf{c} &= e^{2t}\begin{pmatrix} \cosh(4t) - \frac{1}{4}\sinh(4t) & \frac{5}{4}\sinh(4t) \\ \frac{3}{4}\sinh(4t) & \cosh(4t) + \frac{1}{4}\sinh(4t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{2t}\begin{pmatrix} \cosh(4t) - \frac{1}{4}\sinh(4t) \\ \frac{3}{4}\sinh(4t) \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} \frac{5}{4}\sinh(4t) \\ \cosh(4t) + \frac{1}{4}\sinh(4t) \end{pmatrix} . \end{aligned}$$

(9) [10] Find the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4 + 17\mathrm{D}^2 + 16$.

Solution from Green Function. The operator $\mathrm{D}^4 + 17\mathrm{D}^2 + 16$ has characteristic polynomial

$$p(s) = s^4 + 17s^2 + 16 = (s^2+1)(s^2+16) \,.$$

We have the partial-fraction identity

$$\frac{1}{p(s)} = \frac{1}{(s^2+1)(s^2+16)} = \frac{\frac{1}{15}}{s^2+1} + \frac{-\frac{1}{15}}{s^2+16} \,.$$

Referring to the table on the last page, item 2 with $a = 0$ and $b = 1$ and with $a = 0$ and $b = 4$ shows that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t) = \sin(t) \,, \qquad \mathcal{L}^{-1}\left[\frac{4}{s^2+4^2}\right](t) = \sin(4t) \,.$$

Therefore the Green function for the operator $\mathrm{D}^4 + 17\mathrm{D}^2 + 16$ is

$$\begin{aligned} g(t) = \mathcal{L}^{-1}\left[\frac{1}{p(s)}\right](t) &= \tfrac{1}{15}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t) - \tfrac{1}{60}\mathcal{L}^{-1}\left[\frac{4}{s^2+4^2}\right](t) \\ &= \tfrac{1}{15}\sin(t) - \tfrac{1}{60}\sin(4t) \,. \end{aligned}$$

Because we see the characteristic polynomial as

$$p(s) = s^4 + 0\cdot s^3 + 17\cdot s^2 + 0\cdot s + 16 \,,$$

the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4 + 10\mathrm{D}^2 + 9$ is given by

$$\begin{aligned} N_3(t) &= g(t) &&= \tfrac{1}{15}\sin(t) - \tfrac{1}{60}\sin(4t)\,, \\ N_2(t) &= N_3'(t) + 0\cdot g(t) &&= \tfrac{1}{15}\cos(t) - \tfrac{1}{15}\cos(4t)\,, \\ N_1(t) &= N_2'(t) + 17\cdot g(t) &&= -\tfrac{1}{15}\sin(t) + \tfrac{4}{15}\sin(4t) + 17\big(\tfrac{1}{15}\sin(t) - \tfrac{1}{60}\sin(4t)\big)\,, \\ & &&= \tfrac{16}{15}\sin(t) - \tfrac{1}{60}\sin(4t)\,, \\ N_0(t) &= N_1'(t) + 0\cdot g(t) &&= \tfrac{16}{15}\cos(t) - \tfrac{1}{15}\cos(4t)\,. \end{aligned}$$

Solution from General Initial-Value Problem. For the operator $\mathrm{D}^4+17\mathrm{D}^2+16$ the general initial-value problem for initial-time 0 is

$$y'''' + 17y'' + 16y = 0\,, \qquad y(0) = y_0\,, \quad y'(0) = y_1\,, \quad y''(0) = y_2\,, \quad y'''(0) = y_3\,.$$

Its characteristic polynomial is

$$p(z) = z^4 + 17z^2 + 16 = (z^2+1)(z^2+16) = (z^2+1)(z^2+4^2)\,,$$

which has roots i, $-i$, $i4$ and $-i4$. Therefore a real general solution is

$$y(t) = c_1\cos(t) + c_2\sin(t) + c_3\cos(4t) + c_4\sin(4t)\,.$$

Because

$$\begin{aligned} y'(t) &= -c_1\sin(t) + c_2\cos(t) - 4c_3\sin(4t) + 4c_4\cos(4t)\,, \\ y''(t) &= -c_1\cos(t) - c_2\sin(t) - 16c_3\cos(4t) - 16c_4\sin(4t)\,, \\ y'''(t) &= c_1\sin(t) - c_2\cos(t) + 64c_3\sin(4t) - 64c_4\cos(4t)\,, \end{aligned}$$

the general initial conditions yield the linear algebraic system

$$\begin{aligned} y_0 &= y(0) = c_1\cos(0) + c_2\sin(0) + c_3\cos(0) + c_4\sin(0) = c_1 + c_3\,. \\ y_1 &= y'(0) = -c_1\sin(0) + c_2\cos(0) - 4c_3\sin(0) + 4c_4\cos(0) = c_2 + 4c_4\,, \\ y_2 &= y''(0) = -c_1\cos(0) - c_2\sin(0) - 16c_3\cos(0) - 16c_4\sin(0) = -c_1 - 16c_3\,, \\ y_3 &= y'''(t) = c_1\sin(0) - c_2\cos(0) + 64c_3\sin(0) - 64c_4\cos(0) = -c_2 - 64c_4\,. \end{aligned}$$

This decouples into the two systems

$$\begin{aligned} y_0 &= c_1 + c_3\,, \qquad & y_1 &= c_2 + 4c_4\,, \\ y_2 &= -c_1 - 16c_3\,, \qquad & y_3 &= -c_2 - 64c_4\,. \end{aligned}$$

The solutions of these systems are

$$\begin{aligned} c_1 &= \frac{16y_0 + y_2}{15}\,, \qquad & c_2 &= \frac{16y_1 + y_3}{15}\,, \\ c_3 &= -\frac{y_0 + y_2}{15}\,, \qquad & c_4 &= -\frac{y_1 + y_3}{60}\,. \end{aligned}$$

Therefore the solution of the general initial-value problem is

$$\begin{aligned} y &= \frac{16y_0 + y_2}{15}\cos(t) + \frac{16y_1 + y_3}{15}\sin(t) - \frac{y_0 + y_2}{15}\cos(4t) - \frac{y_1 + y_3}{60}\sin(4t) \\ &= y_0\big(\tfrac{16}{15}\cos(t) - \tfrac{1}{15}\cos(4t)\big) + y_1\big(\tfrac{16}{15}\sin(t) - \tfrac{1}{60}\sin(4t)\big) \\ &\quad + y_2\big(\tfrac{1}{15}\cos(t) - \tfrac{1}{15}\cos(4t)\big) + y_3\big(\tfrac{1}{15}\sin(t) - \tfrac{1}{60}\sin(4t)\big)\,. \end{aligned}$$

We can read off from this that the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4 + 10\mathrm{D}^2 + 9$ is

$$N_0(t) = \tfrac{16}{15}\cos(t) - \tfrac{1}{15}\cos(4t)\,, \qquad N_1(t) = \tfrac{16}{15}\sin(t) - \tfrac{1}{60}\sin(4t)\,,$$
$$N_2(t) = \tfrac{1}{15}\cos(t) - \tfrac{1}{15}\cos(4t)\,, \qquad N_3(t) = \tfrac{1}{15}\sin(t) - \tfrac{1}{60}\sin(4t)\,.$$

(10) [8] Compute the Laplace transform of $f(t) = u(t-5)\,e^{-3t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\int_0^T e^{-st}u(t-5)\,e^{-3t}\,\mathrm{d}t = \lim_{T\to\infty}\int_5^T e^{-(s+3)t}\,\mathrm{d}t\,.$$

- When $s \leq -3$ we have for every $T > 5$

$$\int_5^T e^{-(s+3)t}\,\mathrm{d}t \geq \int_5^T \mathrm{d}t = T - 5\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

- When $s > -3$ we have for every $T > 5$

$$\int_5^T e^{-(s+3)t}\,\mathrm{d}t = -\frac{e^{-(s+3)t}}{s+3}\bigg|_5^T = -\frac{e^{-(s+3)T}}{s+3} + \frac{e^{-(s+3)5}}{s+3}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[-\frac{e^{-(s+3)T}}{s+3} + \frac{e^{-(s+3)5}}{s+3}\right] = \frac{e^{-(s+3)5}}{s+3} \qquad \text{for } s > -3\,.$$

Therefore the definition of the Laplace transform gives

$$\mathcal{L}[f](s) = \begin{cases} \dfrac{e^{-(s+3)5}}{s+3} & \text{for } s > -3\,, \\ \text{undefined} & \text{for } s \leq -3\,. \end{cases}$$

(11) [10] Consider the following MATLAB commands.

```
>> syms t x(t) s X
>> f = t^2 + heaviside(t − 2)*(4 − t^2);
>> diffeqn = diff(x, 2) + 4*diff(x, 1) + 20*x(t) == f;
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, ...
            [laplace(x(t), t, s), x(0), subs(diff(x(t), t), t, 0)], [X, 2, −3]);
>> xtrans = simplify(solve(algeqn, X));
>> x = ilaplace(xtrans, s, t)
```

(a) [2] Give the initial-value problem for $x(t)$ that is being solved.

(b) [8] Find the Laplace transform $X(s)$ of the solution $x(t)$. (Just solve for $X(s)$! DO NOT take the inverse Laplace transform of $X(s)$ to find $x(t)$!)

You may refer to the table on the last page.

Solution (a). The initial-value problem for $x(t)$ that is being solved is

$$x'' + 4x' + 20x = f(t)\,, \qquad x(0) = 2\,, \quad x'(0) = -3\,,$$

where the forcing $f(t)$ can be expressed either as the piecewise-defined function

$$f(t) = \begin{cases} t^2 & \text{for } 0 \le t < 2\,, \\ 4 & \text{for } 2 \le t\,. \end{cases}$$

or in terms of the unit step function as

$$f(t) = t^2 + u(t-2)(4-t^2)\,.$$

Solution (b). The Laplace transform of the differential equation is

$$\mathcal{L}[x''](s) + 4\mathcal{L}[x'](s) + 20\mathcal{L}[x](s) = \mathcal{L}[f](s)\,,$$

while the initial conditions give

$$\begin{aligned} \mathcal{L}[x](s) &= X(s)\,, \\ \mathcal{L}[x'](s) &= s\,\mathcal{L}[x](s) - x(0) = s\,X(s) - 2\,, \\ \mathcal{L}[x''](s) &= s\,\mathcal{L}[x'](s) - x'(0) = s^2 X(s) - 2s + 3\,. \end{aligned}$$

Therefore the Laplace transform of the initial-value problem is

$$\big(s^2 X(s) - 2s + 3\big) + 4\big(s\,X(s) - 2\big) + 20X(s) = \mathcal{L}[f](s)\,.$$

This simplifies to

$$(s^2 + 4s + 20)X(s) - 2s - 5 = \mathcal{L}[f](s)\,,$$

whereby

$$X(s) = \frac{1}{s^2 + 4s + 20}\big(2s + 5 + \mathcal{L}[f](s)\big)\,.$$

To compute $\mathcal{L}[f](s)$, we write $f(t)$ as

$$f(t) = t^2 + u(t-2)(4-t^2) = t^2 + u(t-2)j(t-2)\,,$$

where upon setting $j(t-2) = 4 - t^2$, we see by the shifty step method that

$$j(t) = 4 - (t+2)^2 = 4 - t^2 - 4t - 4 = -t^2 - 4t\,.$$

Referring to the table on the last page, item 1 with $a = 0$ and $n = 1$, and with $a = 0$ and $n = 2$ shows that

$$\mathcal{L}[t](s) = \frac{1}{s^2}\,, \qquad \mathcal{L}[t^2](s) = \frac{2}{s^3}\,,$$

whereby item 6 with $c = 2$ and $j(t) = -t^2 - 4t$ shows that

$$\begin{aligned} \mathcal{L}\big[u(t-2)j(t-2)\big](s) = e^{-2s}\mathcal{L}[j](s) &= -e^{-2s}\mathcal{L}[t^2 + 4t](s) \\ &= -e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2}\right)\,. \end{aligned}$$

Therefore

$$\mathcal{L}[f](s) = \mathcal{L}\big[t^2 + u(t-2)j(t-2)\big](s)$$
$$= \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2}\right).$$

Upon placing this result into the expression for $X(s)$ found earlier, we obtain

$$X(s) = \frac{1}{s^2 + 4s + 20}\left(2s + 5 + \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2}\right)\right).$$

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[Y(s)](t)$ of the function

$$Y(s) = e^{-2s}\,\frac{s+8}{s^2 - 8s + 25}.$$

You may refer to the table on the last page.

Solution. Referring to the table on the last page, item 6 with $c = 2$ shows that

$$\mathcal{L}^{-1}\big[e^{-2s}\,J(s)\big] = u(t-2)j(t-2)\,, \qquad \text{where} \qquad j(t) = \mathcal{L}^{-1}[J(s)](t)\,.$$

We apply this formula to

$$J(s) = \frac{s+8}{s^2 - 8s + 25}.$$

Because $s^2 - 8s + 25 = (s-4)^2 + 3^2$, we have the partial fraction identity

$$J(s) = \frac{s+8}{s^2 - 8s + 25} = \frac{(s-4) + 12}{(s-4)^2 + 3^2} = \frac{s-4}{(s-4)^2 + 3^2} + \frac{12}{(s-4)^2 + 3^2}.$$

Referring to the table on the last page, items 2 and 3 with $a = 4$ and $b = 3$ show that

$$\mathcal{L}^{-1}\left[\frac{s-4}{(s-4)^2 + 3^2}\right] = e^{4t}\cos(3t)\,, \qquad \mathcal{L}^{-1}\left[\frac{3}{(s-4)^2 + 3^2}\right] = e^{4t}\sin(3t)\,.$$

The above formulas and the linearity of the inverse Laplace transform yield

$$j(t) = \mathcal{L}^{-1}[J(s)](t) = \mathcal{L}^{-1}\left[\frac{s+8}{s^2 - 8s + 25}\right](t)$$
$$= \mathcal{L}^{-1}\left[\frac{s-4}{(s-4)^2 + 3^2} + \frac{12}{(s-4)^2 + 3^2}\right](t)$$
$$= \mathcal{L}^{-1}\left[\frac{s-4}{(s-4)^2 + 3^2}\right](t) + 4\mathcal{L}^{-1}\left[\frac{3}{(s-4)^2 + 3^2}\right](t)$$
$$= e^{4t}\cos(3t) + 4e^{4t}\sin(3t)\,.$$

Therefore

$$\mathcal{L}^{-1}\big[Y(s)\big](t) = \mathcal{L}^{-1}[e^{-2s}J(s)](t)$$
$$= u(t-2)j(t-2)$$
$$= u(t-2)\left(e^{4(t-2)}\cos\big(3(t-2)\big) + 4e^{4(t-2)}\sin\big(3(t-2)\big)\right).$$

Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[j'(t)](s) = sJ(s) - j(0) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s),\ c \geq 0,$$
and u is the unit step function.

$$\mathcal{L}[\delta(t-c)j(t)](s) = e^{-cs}j(c) \qquad \text{where } c \geq 0 \text{ and } \delta \text{ is the unit impulse.}$$