Third In-Class Exam
Math 246, Professor David Levermore
Thursday, 14 November 2019

Your Name:____

UMD SID:____

Discussion Instructor (circle one): Sam Potter Nathan Yu David Russell
Discussion Time (circle one): 9:00 11:00 12:00

No books, notes, calculators, or any electronic devices. If you need more space to answer a problem then use the back of one of these pages. **Do not separate the pages!** Indicate where the answer to each part of each problem is located. Cross out work that you do not want considered. **Your reasoning must be given for full credit!** Good luck!

Please copy and sign the University Honor Pledge below.
Please print your name on each page.

University Honor Pledge: *I pledge on my honor that I have not given or received any unauthorized assistance on this examination.* ____

____

____

Signature:____

Problem 1:	________/6	Problem 2:	________/10
Problem 3:	________/6	Problem 4:	________/10
Problem 5:	________/8	Problem 6:	________/8
Problem 7:	________/8	Problem 8:	________/8
Problem 9:	________/10	Problem 10:	________/8
Problem 11:	________/10	Problem 12:	________/8

Total Score: ________/100 Grade: ________

Name:____

(1) [6] Recast the ordinary differential equation $y'''' - e^y y''' + e^t y'' - \sin(t + y') = 0$ as a first-order system of ordinary differential equations.

(2) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^2 \\ -1 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

(a) [2] Compute the Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{C}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\mathbf{x}' = \mathbf{C}(t)\mathbf{x}$ wherever $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) [2] Give a general solution to the system found in part (b).

(d) [3] Compute the Green matrix associated with the system found in part (b).

Name:____

(3) [6] Given that 2 is an eigenvalue of the matrix

$$\mathbf{C} = \begin{pmatrix} 4 & 0 & -4 \\ 0 & 3 & 3 \\ 2 & 2 & 4 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{C}$ associated with 2.

(4) [10] Solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Name:____________________________________

(5) [8] Two interconnected tanks are filled with brine (salt water). At $t = 0$ the first tank contains 17 liters and the second contains 28 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 21 grams of salt in the first tank and 14 grams in the second.

(a) [6] Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(b) [2] Give the interval of definition for the solution of this initial-value problem.

(6) [8] A 4×4 matrix $\mathbf{K}$ has the eigenpairs

$$\left(0, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right), \qquad \left(1, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}\right), \qquad \left(4, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\right), \qquad \left(9, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\right),$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $e^{t\mathbf{K}} = \mathbf{V}e^{t\mathbf{D}}\mathbf{V}^{-1}$. (You do not have to compute either $\mathbf{V}^{-1}$ or $e^{t\mathbf{K}}$!)

(b) Give a fundamental matrix for the system $\mathbf{x}' = \mathbf{K}\mathbf{x}$.

Name:____

(7) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 & -4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(8) [8] Find a real general solution of the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Name:____

(9) [10] Find the natural fundamental set of solutions associated with the initial-time 0 for the operator $\mathrm{D}^4 + 17\mathrm{D}^2 + 16$.

(10) [8] Compute the Laplace transform of $f(t) = u(t-5)\, e^{-3t}$ from its definition. (Here u is the unit step function.)

Name:______________________________________

(11) [10] Consider the following MATLAB commands.

```
>> syms t x(t) s X
>> f = t^2 + heaviside(t − 2)*(4 − t^2);
>> diffeqn = diff(x, 2) + 4*diff(x, 1) + 20*x(t) == f;
>> eqntrans = laplace(diffeqn, t, s);
>> algeqn = subs(eqntrans, ...
        [laplace(x(t), t, s), x(0), subs(diff(x(t), t), t, 0)], [X, 2, −3]);
>> xtrans = simplify(solve(algeqn, X));
>> x = ilaplace(xtrans, s, t)
```

(a) [2] Give the initial-value problem for $x(t)$ that is being solved.

(b) [8] Find the Laplace transform $X(s)$ of the solution $x(t)$. (Just solve for $X(s)$! DO NOT take the inverse Laplace transform of $X(s)$ to find $x(t)$!)

You may refer to the table on the last page.

(12) [8] Find the inverse Laplace transform $\mathcal{L}^{-1}[Y(s)](t)$ of the function

$$Y(s) = e^{-2s}\,\frac{s+8}{s^2 - 8s + 25}\,.$$

You may refer to the table on the last page.

Name:____

Workspace (Give the number of the problem being worked!)

Table of Laplace Transforms

$\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$	for $s > a$.
$\mathcal{L}[e^{at}\cos(bt)](s) = \dfrac{s-a}{(s-a)^2+b^2}$	for $s > a$.
$\mathcal{L}[e^{at}\sin(bt)](s) = \dfrac{b}{(s-a)^2+b^2}$	for $s > a$.
$\mathcal{L}[j'(t)](s) = sJ(s) - j(0)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[e^{at} j(t)](s) = J(s-a)$	where $J(s) = \mathcal{L}[j(t)](s)$.
$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s)$	where $J(s) = \mathcal{L}[j(t)](s)$, $c \geq 0$, and u is the unit step function.
$\mathcal{L}[\delta(t-c)j(t)](s) = e^{-cs}j(c)$	where $c \geq 0$ and δ is the unit impulse.