Second In-Class Exam Solutions
Math 246, Professor David Levermore
Thursday, 17 October 2019

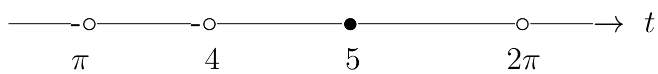
(1) [4] Give the interval of definition for the solution of the initial-value problem

$$x''' - \frac{\cos(3t)}{t^2-16}\,x' + \frac{e^t}{\sin(t)}\,x = \frac{1}{1+t^2}\,, \qquad x(5) = x'(5) = x''(5) = -2\,.$$

Solution. The equation is linear and is already in normal form. Notice the following.

◇ The coefficient of x' is undefined at $t = \pm 4$ and is continuous elsewhere.
◇ The coefficient of x is undefined at $t = n\pi$ for every integer n and is continuous elsewhere.
◇ The forcing is continuous everywhere.
◇ The initial time is $t = 5$.

Plotting these points on a time-line near the initial time $t = 5$ gives



Therefore the interval of definition is $(4, 2\pi)$ because:

- the initial time $t = 5$ is in $(4, 2\pi)$;
- all the coefficients and the forcing are continuous over $(4, 2\pi)$;
- the coefficient of x' is undefined at $t = 4$;
- the coefficient of x is undefined at $t = 2\pi$.

Remark. All four reasons must be given for full credit.

○ The first two reasons are why a (unique) solution exists over the interval $(4, 2\pi)$.
○ The last two reasons are why this solution does not exist over a larger interval.

(2) [12] The functions t and t^2 are a fundamental set of solutions to $t^2y'' - 2ty' + 2y = 0$ over $t > 0$.

(a) [8] Solve the general initial-value problem

$$t^2y'' - 2ty' + 2y = 0\,, \qquad y(1) = y_0\,, \quad y'(1) = y_1\,.$$

(b) [4] Find the associated natural fundamental set of solutions to $t^2y'' - 2ty' + 2y = 0$.

Solution (a). Because we are given that t and t^2 is a fundamental set of solutions to $t^2y'' - 2ty' + 2y = 0$ over $t > 0$, a general solution is $y(t) = c_1t + c_2t^2$. Because $y'(t) = c_1 + 2c_2t$, the initial conditions imply

$$y_0 = y(1) = c_1 + c_2\,, \qquad\qquad y_1 = y'(1) = c_1 + 2c_2\,.$$

We solve these equations to obtain

$$c_1 = 2y_0 - y_1\,, \qquad\qquad c_2 = y_1 - y_0\,.$$

Therefore the solution to the general initial-value problem is

$$y(t) = (2y_0 - y_1)t + (y_1 - y_0)t^2\,.$$

Solution (b). The solution found in part (a) can be written as

$$y(t) = y_0(2t - t^2) + y_1(t^2 - t)\,.$$

We can read off from this that the associated natural fundamental set of solutions is

$$N_0(t) = 2t - t^2\,, \qquad N_1(t) = t^2 - t\,.$$

(3) [4] Suppose that $Z_1(t)$, $Z_2(t)$, $Z_3(t)$, and $Z_4(t)$ solve the differential equation

$$z'''' + 3z''' + \sin(2t)z' + e^t z' + 6z = 0\,,$$

Suppose we know that $\mathrm{Wr}[Z_1, Z_2, Z_3, Z_4](0) = 5$. Find $\mathrm{Wr}[Z_1, Z_2, Z_3, Z_4](t)$.

Solution. The Abel Theorem says that $w(t) = \mathrm{Wr}[Z_1, Z_2, Z_3, Z_4](t)$ satisfies

$$w' + 3w = 0\,.$$

We see that $w(t) = ce^{-3t}$ for some c. Because $w(t)$ satisfies the initial condition

$$w(0) = \mathrm{Wr}[Z_1, Z_2, Z_3, Z_4](0) = 5\,,$$

we have $w(0) = ce^{-3\cdot 0} = 5$, whereby $c = 5$. Therefore $w(t) = 5e^{-3t}$, which shows that

$$\mathrm{Wr}[Z_1, Z_2, Z_3, Z_4](t) = 5e^{-3t}\,.$$

(4) [12] Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $-3 + i2$, $-3 + i2$, $-3 - i2$, $-3 - i2$, -5, -5, -5, 0, 0.

(a) [2] Give the order of L.
(b) [7] Give a real general solution of the homogeneous equation $\mathrm{L}u = 0$.
(c) [3] Give the degree d, characteristic $\mu + i\nu$, and multiplicity m for the forcing of the nonhomogeneous equation $\mathrm{L}v = t^4 e^{-3t}\sin(2t)$.

Solution (a). Because 9 roots are listed, the degree of the characteristic polynomial must be 9, whereby the order of L is 9.

Solution (b). A fundamental set of nine real-valued solutions is built as follows.

- ⋄ The conjugate pair of double roots $-3 \pm i2$ contributes

$$e^{-3t}\cos(2t)\,, \quad e^{-3t}\sin(2t)\,, \quad t\,e^{-3t}\cos(2t)\,, \quad \text{and} \quad t\,e^{-3t}\sin(2t)\,.$$

- ⋄ The triple real root -5 contributes

$$e^{-5t}\,, \quad t\,e^{-5t}\,, \qquad \text{and} \qquad t^2 e^{-5t}\,.$$

- ⋄ The double real root 0 contributes

$$1 \qquad \text{and} \qquad t\,.$$

Therefore a real general solution of the homogeneous equation $\mathrm{L}u = 0$ is

$$u = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t) + c_3 t\,e^{-3t}\cos(2t) + c_4 t\,e^{-3t}\sin(2t)$$
$$+ c_5 e^{-5t} + c_6 t\,e^{-5t} + c_7 t^2 e^{-5t} + c_8 + c_9 t\,.$$

Solution (c). The forcing of the nonhomogeneous linear equation $\mathrm{L}v = t^4 e^{-3t}\sin(2t)$ has degree $d = 4$ and characteristic $\mu + i\nu = -3 + i2$. Because the characteristic $\mu + i\nu = -3 + i2$ is listed as a double root of the characteristic polynomial, it has multiplicity $m = 2$. Therefore, we have

$$d = 4\,, \qquad \mu + i\nu = -3 + i2\,, \qquad m = 2\,.$$

(5) [8] What answer will be produced by the following MATLAB commands?

```
>> syms w(t)
>> ode = diff(w,t,2) − diff(w,t) − 12*w == 12*exp(3*t);
>> wSol(t) = dsolve(ode)
```

Solution. The commands ask MATLAB for a real general solution of the equation

$$\mathrm{D}^2 w - \mathrm{D}w - 12w = 12e^{3t}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

While your answer did not have to be given in MATLAB format, MATLAB will produce something equivalent to

```
− 2*exp(3*t) + C1*exp(−3*t) + C2*exp(4*t)
```

This can be seen as follows. This is a *nonhomogeneous* linear equation for $w(t)$ with *constant coefficients*. Its linear differential operator is $\mathrm{L} = \mathrm{D}^2 - \mathrm{D} - 12$. Its characteristic polynomial is

$$p(z) = z^2 - z - 12 = (z+3)(z-4)\,,$$

which has the two real roots -3 and 4. Therefore a real general solution of the associated homogeneous problem is

$$w_H(t) = c_1 e^{-3t} + c_2 e^{4t}\,.$$

The forcing $12e^{3t}$ has degree $d = 0$, characteristic $\mu + i\nu = 3$, and multiplicity $m = 0$. A particular solution $w_P(t)$ can be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Below we show that each of these methods gives the particular solution $w_P(t) = -2e^{3t}$. Therefore a real general solution is

$$w = c_1 e^{-3t} + c_2 e^{4t} - 2\, e^{3t}\,.$$

Up to notational differences, this is the answer that MATLAB produces.

Key Identity Evaluations. Because $m = m + d = 0$, we only need to evaluate the Key Identity at the characteristic $z = \mu + i\nu = 3$. The Key Identity is

$$\mathrm{L}(e^{zt}) = (z^2 - z - 12)\cdot e^{zt}\,.$$

When this is evaluated at $z = 3$ we find

$$\mathrm{L}(e^{3t}) = (3^2 - 3 - 12)\cdot e^{3t} = -6e^{3t}\,.$$

Because the forcing is $12e^{3t}$, we multiply the above equation by -2 to obtain

$$\mathrm{L}(-2e^{3t}) = 12e^{3t}\,.$$

Therefore a particular solution is

$$w_P(t) = -2e^{3t}\,.$$

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the phasor form

$$f(t) = \alpha e^{\mu t} \cos(\nu t) + \beta e^{\mu t} \sin(\nu t) = e^{\mu t} \operatorname{Re}\big((\alpha - i\beta) e^{i\nu t}\big) ,$$

this formula gives the particular solution

$$w_P(t) = t^m e^{\mu t} \operatorname{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)} \, e^{i\nu t} \right) .$$

For this problem the forcing $f(t) = 12e^{3t}$ is already in phasor form with phasor $\alpha - i\beta = 12$ and characteristic $\mu + i\nu = 3$. Because the characteristic polynomial is $p(z) = z^2 - z - 12$ and $m = 0$, we have

$$p^{(m)}(\mu + i\nu) = p(3) = 3^2 - 3 - 12 = -6 .$$

Therefore the particular solution becomes

$$w_P(t) = e^{3t} \, \frac{12}{-6} = -2e^{3t} .$$

Undetermined Coefficients. Because $m = m + d = 0$ and $\mu + i\nu = 3$, there is a particular solution in the form

$$w_P(t) = Ae^{3t} .$$

Because

$$w_P'(t) = 3Ae^{3t} , \qquad w_P''(t) = 9Ae^{3t} ,$$

we see that

$$\begin{aligned} \mathrm{L}w_P(t) &= w_P''(t) - w_P'(t) - 12w_P(t) \\ &= \big[9Ae^{3t}\big] - \big[3Ae^{3t}\big] - 12\big[Ae^{3t}\big] \\ &= (9 - 3 - 12)Ae^{3t} = -6Ae^{3t} . \end{aligned}$$

Setting $\mathrm{L}w_P(t) = -6Ae^{3t} = 12e^{3t}$, we see that $A = -2$. Therefore the particular solution is

$$w_P(t) = -2e^{2t} .$$

(6) [8] Find a particular solution $q_P(t)$ of the equation $q'' - 4q = 8t\, e^{2t}$.

Solution. This is a *nonhomogeneous* linear equation with *constant coefficients*. Its linear differential operator is $\mathrm{L} = \mathrm{D}^2 - 4$. Its characteristic polynomial is

$$p(z) = z^2 - 4 = (z + 2)(z - 2) ,$$

which has two simple real roots -2 and 2. The forcing $8t\, e^{2t}$ has characteristic form with degree $d = 1$ and characteristic $\mu + i\nu = 2$, which has multiplicity $m = 1$. Therefore we can use either Key Identity Evaluations or Undetermined Coefficients to find a particular solution. Both methods give the particular solution

$$q_P(t) = t^2 e^{2t} - \tfrac{1}{2} t\, e^{2t} .$$

Key Identity Evaluations. Because $m = 1$ and $m + d = 2$ we need to evaluate the first and second derivative (with respect to z) of the Key Identity at the characteristic $z = \mu + i\nu = 2$. The Key Identity and its first two derivatives with respect to z are

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= (z^2 - 4) \cdot e^{zt} \,, \\ \mathrm{L}\big(t\, e^{zt}\big) &= (z^2 - 4) \cdot t\, e^{zt} + 2z \cdot e^{zt} \,, \\ \mathrm{L}\big(t^2 e^{zt}\big) &= (z^2 - 4) \cdot t^2 e^{zt} + 2 \cdot 2z \cdot t\, e^{zt} + 2 \cdot e^{zt} \,. \end{aligned}$$

(Notice the 2 in the middle term of the second derivative from the Pascal triangle.) By evaluating the first and second derivative of the Key Identity at $z = \mu + i\nu = 2$ we obtain

$$\mathrm{L}\big(t\, e^{2t}\big) = 4e^{2t} \,, \qquad \mathrm{L}\big(t^2 e^{2t}\big) = 8t\, e^{2t} + 2e^{2t} \,.$$

By subtracting half of the first equation from the second we obtain

$$\mathrm{L}\big(t^2 e^{2t} - \tfrac{1}{2} t\, e^{2t}\big) = 8t\, e^{2t} \,.$$

Therefore a particular solution of $\mathrm{L}q = 8t\, e^{2t}$ is

$$q_P(t) = t^2 e^{2t} - \tfrac{1}{2} t\, e^{2t} \,.$$

Undetermined Coefficients. Because $m = 1$, $m + d = 2$, and $\mu + i\nu = 2$, there is a particular solution in the form

$$q_P(t) = (A_0 t^2 + A_1 t)\, e^{2t} \,.$$

Because

$$\begin{aligned} q_P'(t) &= 2(A_0 t^2 + A_1 t)\, e^{2t} + (2A_0 t + A_1)\, e^{2t} \\ &= \big(2A_0 t^2 + (2A_0 + 2A_1)t + A_1\big)\, e^{2t} \,, \\ q_P''(t) &= 2\big(2A_0 t^2 + (2A_0 + 2A_1)t + A_1\big)\, e^{2t} + \big(4A_0 t + (2A_0 + 2A_1)\big)\, e^{2t} \\ &= \big(4A_0 t^2 + (8A_0 + 4A_1)t + (2A_0 + 4A_1)\big)\, e^{2t} \,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}q_P(t) &= q_P''(t) - 4q_P(t) \\ &= \big(4A_0 t^2 + (8A_0 + 4A_1)t + (2A_0 + 4A_1)\big)\, e^{2t} - 4(A_0 t^2 + A_1 t)\, e^{2t} \\ &= \big(8A_0 t + (2A_0 + 4A_1)\big)\, e^{2t} = 8A_0 t\, e^{2t} + (2A_0 + 4A_1) e^{2t} \,. \end{aligned}$$

By setting $\mathrm{L}q_P(t) = 8t\, e^{2t}$, the linear independence of $t\, e^{2t}$ and e^{2t} implies that

$$8A_0 = 8 \,, \qquad 2A_0 + 4A_1 = 0 \,,$$

which yields $A_0 = 1$ and $A_1 = -\frac{1}{2}$. Therefore a particular solution of $\mathrm{L}q = 8t\, e^{2t}$ is

$$q_P(t) = t^2 e^{2t} - \tfrac{1}{2} t\, e^{2t} \,.$$

(7) [8] Compute the Green function $g(t)$ associated with the differential operator

$$\mathrm{D}^2 + 6\mathrm{D} + 9\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. Because the linear differential operator has constant coefficients, its Green function $g(t)$ satisfies

$$\mathrm{D}^2 g + 6\mathrm{D}g + 9g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The characteristic polynomial is

$$p(z) = z^2 + 6z + 9 = (z+3)^2\,,$$

which has the double root -3. Hence, a general solution of the equation is

$$g(t) = c_1 e^{-3t} + c_2 t\, e^{-3t}\,.$$

The first initial condition implies $0 = g(0) = c_1$, whereby

$$g(t) = c_2 t e^{-3t}\,.$$

Because

$$g'(t) = c_2 e^{-3t} - 3c_2 t\, e^{-3t}\,,$$

the second initial condition implies $1 = g'(0) = c_2$, whereby $c_2 = 1$. Therefore the Green function associated with the differential operator is

$$g(t) = t e^{-3t}\,.$$

(8) [8] Solve the initial-value problem

$$v'' + 6v' + 9v = \frac{9e^{-3t}}{1+t}\,, \qquad v(0) = v'(0) = 0\,.$$

Solution. This is a *nonhomogeneous* linear equation with *constant coefficients*. Because its forcing does *not have characteristic form*, we cannot use either Key Identity Evaluations or Undetermined Coefficients. Because this is an initial-value problem with *homogeneous initial conditions*, we will use the Green function method, which leads directly to the answer.

By the previous problem the Green function for this problem is $g(t) = t\,e^{-3t}$. Because the equation is in normal form, the initial time is 0, and both of the initial values are 0, the solution to this inital-value problem is given by the Green formula

$$\begin{aligned} q(t) = \int_0^t g(t-s)f(s)\,\mathrm{d}s &= \int_0^t (t-s)e^{-3(t-s)}\,\frac{9e^{-3s}}{1+s}\,\mathrm{d}s \\ &= 9e^{-3t}\int_0^t \frac{t-s}{1+s}\,\mathrm{d}s \\ &= 9t\,e^{-3t}\int_0^t \frac{1}{1+s}\,\mathrm{d}s - 8e^{-3t}\int_0^t \frac{s}{1+s}\,\mathrm{d}s \\ &= 9t\,e^{-3t}\log(1+t) - 9e^{-3t}\big(t - \log(1+t)\big)\,. \end{aligned}$$

Remark. The last integral above can be done by using either the substitution $u = 1+s$, integration by parts, or the identity

$$\frac{s}{1+s} = 1 - \frac{1}{1+s}\,.$$

Remark. Notice that the interval of definition for this solution is $(-1, \infty)$, which is a fact that could have been read off directly from the initial-value problem beforehand.

Remark. This problem can also be solved by the general Green function method. However that approach is not as efficient because it does not use the fact the Green function $g(t)$ was already computed in the solution of the preceeding problem. The integrals end up being the same.

Remark. This problem can also be solved by using variation of parameters. However that approach is not as efficient because it does not directly solve the initial-value problem. Rather, it yields a general solution after which the parameters c_1 and c_2 in it must be determined to satisfy the initial conditions.

(9) [10] The functions $1+2t$ and t^2 are solutions of the homogeneous equation

$$(1+t)t\,x'' - (1+2t)x' + 2x = 0 \qquad \text{over } t > 0\,.$$

(You do not have to check that this is true!)

(a) [3] Show that these functions are linearly independent.

(b) [7] Give a general solution of the nonhomogeneous equation

$$(1+t)t\,y'' - (1+2t)y' + 2y = \frac{8t\,(1+t)^2}{1+2t} \qquad \text{over } t > 0\,.$$

Solution (a). The Wronskian of $1+2t$ and t^2 is

$$\mathrm{Wr}[1+2t, t^2](t) = \det\begin{pmatrix} 1+2t & t^2 \\ 2 & 2t \end{pmatrix} = (1+2t)2t - 2t^2 = 2t + 2t^2 = 2(1+t)t\,.$$

Because $\mathrm{Wr}[1+2t, t^2](t) \neq 0$ for $t > 0$, the functions $1+2t$ and t^2 are linearly independent.

Solution (b). The *nonhomogeneous* equation for $y(t)$ has *variable coefficients*, so we must use either the variation of parameters method or the general Green function method to solve it. Because we seek a general solution, neither method is favored. To apply either method we must first bring the equation into its normal form,

$$y'' - \frac{1+2t}{t}\,y' + \frac{2}{t}\,y = \frac{8(1+t)}{1+2t} \qquad \text{over } t > 0\,.$$

Because $1+2t$ and t^2 are linearly independent, they constitute a fundamental set of solutions to the associated homogeneous equation.

Variation of Parameters. Because $1+2t$ and t^2 constitute a fundamental set of solutions to the associated homogeneous equation, we seek a general solution of the nonhomogeneous equation in the form

$$y(t) = (1+2t)u_1(t) + t^2 u_2(t)\,,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$\begin{aligned} (1+2t)u_1'(t) + t^2 u_2'(t) &= 0\,, \\ 2u_1'(t) + 2t u_2'(t) &= \frac{8(1+t)}{1+2t}\,. \end{aligned}$$

The solution of this system is

$$u_1'(t) = -\frac{4t}{1+2t}\,, \qquad u_2'(t) = \frac{4}{t}\,.$$

Integrate these equations over $t > 0$ to obtain

$$u_1(t) = c_1 - 2t + \log(1+2t)\,, \qquad u_2(t) = c_2 + 4\log(t)\,.$$

Therefore a general solution of the nonhomogeneous equation over $t > 0$ is

$$\begin{aligned} y(t) &= (1+2t)u_1(t) + t^2 u_2(t) \\ &= (1+2t)\big(c_1 - 2t + \log(1+2t)\big) + t^2\big(c_2 + 4\log(t)\big) \\ &= (1+2t)c_1 + t^2 c_2 + (1+2t)\big(\log(1+2t) - 2t\big) + 4t^2\log(t)\,. \end{aligned}$$

Remark. The integration of $u_1'(t)$ above can be done by using either the substitution $u = 1 + 2t$, integration by parts, or the identity

$$\frac{4t}{1+2t} = 2 - \frac{2}{1+2t}\,.$$

Remark. Another way to find $u_1'(t)$ and $u_2'(t)$ is to use the formulas

$$u_1'(t) = -\frac{Y_2(t)\,f(t)}{\mathrm{Wr}[Y_1, Y_2](t)}\,, \qquad u_2'(t) = \frac{Y_1(t)\,f(t)}{\mathrm{Wr}[Y_1, Y_2](t)}\,,$$

with $Y_1(t) = 1 + 2t$, $Y_2(t) = t^2$, and $f(t) = 8(1+t)/(1+2t)$. They yield

$$\begin{aligned} u_1'(t) &= -\frac{t^2}{2(1+t)t}\,\frac{8(1+t)}{1+2t} = -\frac{4t}{1+2t}\,, \\ u_2'(t) &= \frac{1+2t}{2(1+t)t}\,\frac{8(1+t)}{1+2t} = \frac{4}{t}\,. \end{aligned}$$

This approach requires knowing two formulas. The General Green Function method shown next requires knowing just one formula.

General Green Function. The Green function $G(t,s)$ is given by

$$G(t,s) = \frac{1}{\mathrm{Wr}[1+2s, s^2](s)}\,\det\begin{pmatrix} 1+2s & s^2 \\ 1+2t & t^2 \end{pmatrix} = \frac{t^2(1+2s) - (1+2t)s^2}{2(1+s)s}\,.$$

The Green Formula then yields the particular solution

$$\begin{aligned} y_P(t) = \int_1^t G(t,s)\,f(s)\,\mathrm{d}s &= \int_1^t \frac{t^2(1+2s) - (1+2t)s^2}{2(1+s)s}\,\frac{8(1+s)}{1+2s}\,\mathrm{d}s \\ &= 4t^2 \int_1^t \frac{1}{s}\,\mathrm{d}s - (1+2t)\int_1^t \frac{4s}{1+2s}\,\mathrm{d}s \\ &= 4t^2 \log(t) - (1+2t)\left(2t - 2 - \log\left(\frac{1+2t}{3}\right)\right)\,. \end{aligned}$$

A general solution of the nonhomogeneous equation over $t > 0$ is thereby

$$y(t) = c_1(1+2t) + c_2 t^2 + 4t^2\log(t) - (1+2t)\left(2t - 2 - \log\left(\frac{1+2t}{3}\right)\right)\,.$$

Remark. The last integral above can be done by using either the substitution $u = 1 + 2s$, integration by parts, or the identity

$$\frac{4s}{1+2s} = 2 - \frac{2}{1+2s}\,.$$

Remark. Because the integrands are both continuous except at $s = -\frac{1}{2}$ and $s = 0$, and because we want our solution to be defined for every $t > 0$, the lower endpoint of integration in the Green Formula can be any $t_I > 0$. We took $t_I = 1$ because it simplified the evaluation of the first primitive. Had we been asked to solve an initial-value problem then we would have taken t_I to be the initial time. For any $t_I > 0$ the resulting particular solution would satisfy

$$y_P(t_I) = y_P'(t_I) = 0\,.$$

Remark. Notice that the general solutions produced by the Variation of Parameters and General Green Function methods differ because they are built from different particular solutions. If we replace the c_2 in these first of these general solutions by $c_2 + 2 - \log(3)$ then we get the second.

(10) [8] Give a real general solution of the equation

$$\mathrm{D}^2 u - 4\mathrm{D}u + 20u = 4\cos(2t) - 3\sin(2t)\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a *nonhomogeneous* linear equation with *constant coefficients*. Its linear differential operator is $\mathrm{L} = \mathrm{D}^2 - 4\mathrm{D} + 20$. Its characteristic polynomial is

$$p(z) = z^2 - 4z + 20 = (z-2)^2 + 4^2\,,$$

which has the conjugate pair of roots $2 \pm i4$. The forcing $4\cos(2t) - 3\sin(2t)$ has characteristic form with degree $d = 0$ and characteristic $\mu + i\nu = i2$, which has multiplicity $m = 0$. Therefore we can use either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients to find a particular solution. Each of these methods gives the real particular solution

$$v_P(t) = \tfrac{1}{8}\cos(2t) - \tfrac{1}{4}\sin(2t)\,.$$

Therefore a real general solution is

$$v(t) = c_1 e^{2t}\cos(4t) + c_2 e^{2t}\sin(4t) + \tfrac{1}{8}\cos(2t) - \tfrac{1}{4}\sin(2t)\,.$$

Key Identity Evaluations. Because $m = m + d = 0$, we only need to evaluate the Key Identity at the characteristic $z = \mu + i\nu = i2$. The Key Identity is

$$\mathrm{L}(e^{zt}) = (z^2 - 4z + 20)\cdot e^{zt}\,.$$

When this is evaluated at $z = i2$ we find

$$\mathrm{L}(e^{i2t}) = \big((i2)^2 - 4\cdot(i2) + 20\big)\cdot e^{i2t} = (-4 - i8 + 20)e^{i2t} = (16 - i8)e^{i2t}\,.$$

Because the forcing has the phasor form

$$4\cos(2t) - 3\sin(2t) = \mathrm{Re}\big((4 + i3)e^{i2t}\big)\,,$$

we multiply the previous equation by $4 + i3$ and divide by $16 - i8$ to obtain

$$\mathrm{L}\left(\frac{4 + i3}{16 - i8}\,e^{i2t}\right) = (4 + i3)e^{i2t}\,.$$

The real part of this equation gives the particular solution

$$\begin{aligned} v_P(t) = \ &\mathrm{Re}\left(\frac{4+i3}{16-i8}\,e^{i2t}\right) = \tfrac{1}{8}\,\mathrm{Re}\left(\frac{4+i3}{2-i}\,e^{i2t}\right) = \tfrac{1}{8}\,\mathrm{Re}\left(\frac{4+i3}{2-i}\,\frac{2+i}{2+i}\,e^{i2t}\right) \\ &= \tfrac{1}{40}\,\mathrm{Re}\big((4+i3)\,(2+i)\,e^{i2t}\big) = \tfrac{1}{40}\,\mathrm{Re}\big((5+i10)\,e^{i2t}\big) = \tfrac{1}{8}\,\mathrm{Re}\big((1+i2)\,e^{i2t}\big) \\ &= \tfrac{1}{8}\,\mathrm{Re}\Big((1+i2)\,\big(\cos(2t) + i\sin(2t)\big)\Big) = \tfrac{1}{8}\cos(2t) - \tfrac{1}{4}\sin(2t)\,. \end{aligned}$$

Zero Degree Formula. For a forcing $f(t)$ with degree $d = 0$, characteristic $\mu + i\nu$, and multiplicity m that has the phasor form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t) = e^{\mu t}\,\mathrm{Re}\big((\alpha - i\beta)e^{i\nu t}\big)\,,$$

this formula gives the particular solution

$$v_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{\alpha - i\beta}{p^{(m)}(\mu + i\nu)}\,e^{i\nu t}\right)\,.$$

For this problem the forcing has the phasor form

$$f(t) = 4\cos(2t) - 3\sin(2t) = \mathrm{Re}\big((4 + i3)e^{i2t}\big)\,,$$

with characteristic $\mu + i\nu = i2$ and phasor $\alpha - i\beta = 4 + i3$. Because the characteristic polynomial is $p(z) = z^2 - 4z + 20$ and $m = 0$, we have

$$p^{(m)}(\mu + i\nu) = p(i2) = (i2)^2 - 4\cdot(i2) + 20 = -4 - i8 + 20 = 16 - i8\,.$$

Therefore the particular solution becomes

$$\begin{aligned} v_P(t) &= \mathrm{Re}\left(\frac{4 + i3}{16 - i8}\,e^{i2t}\right) = \tfrac{1}{8}\,\mathrm{Re}\left(\frac{4 + i3}{2 - i}\,e^{i2t}\right) = \tfrac{1}{8}\,\mathrm{Re}\left(\frac{4 + i3}{2 - i}\,\frac{2 + i}{2 + i}\,e^{i2t}\right) \\ &= \tfrac{1}{40}\,\mathrm{Re}\big((4 + i3)\,(2 + i)\,e^{i2t}\big) = \tfrac{1}{40}\,\mathrm{Re}\big((5 + i10)\,e^{i2t}\big) = \tfrac{1}{8}\,\mathrm{Re}\big((1 + i2)\,e^{i2t}\big) \\ &= \tfrac{1}{8}\,\mathrm{Re}\big((1 + i2)\,\big(\cos(2t) + i\sin(2t)\big)\big) = \tfrac{1}{8}\cos(2t) - \tfrac{1}{4}\sin(2t)\,. \end{aligned}$$

Undetermined Coefficients. Because $m = m + d = 0$, and $\mu + i\nu = i2$, there is a particular solution in the form

$$v_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$\begin{aligned} v_P'(t) &= -2A\sin(2t) + 2B\cos(2t)\,, \\ v_P''(t) &= -4A\cos(2t) - 4B\sin(2t)\,, \end{aligned}$$

we see that

$$\begin{aligned} \mathrm{L}v_P(t) &= v_P''(t) - 4v_P'(t) + 20v_P(t) \\ &= \big[-4A\cos(2t) - 4B\sin(2t)\big] - 4\big[-2A\sin(2t) + 2B\cos(2t)\big] \\ &\qquad + 20\big[A\cos(2t) + B\sin(2t)\big] \\ &= (16A - 8B)\cos(2t) + (8A + 16B)\sin(2t)\,. \end{aligned}$$

By setting $\mathrm{L}v_P(t) = 4\cos(2t) - 3\sin(2t)$, the linear independence of $\cos(2t)$ and $\sin(2t)$ implies that A and B solve the linear algebric system

$$16A - 8B = 4\,, \qquad 8A + 16B = -3\,.$$

This implies that $A = \frac{1}{8}$ and $B = -\frac{1}{4}$, whereby the particular solution becomes

$$v_P(t) = \tfrac{1}{8}\cos(2t) - \tfrac{1}{4}\sin(2t)\,.$$

(11) [10] The vertical displacement of a spring-mass system is governed by the equation

$$\ddot{h} + 10\dot{h} + 169h = \alpha\cos(\omega t) + \beta\sin(\omega t)\,,$$

where $\alpha \neq 0$, $\beta \neq 0$, and $\omega > 0$. Assume CGS units.

(a) [2] Give the natural frequency and period of the system.

(b) [4] Show the system is under damped and give its damped frequency and period.

(c) [4] Give the steady state solution in its phasor form $\mathrm{Re}(\Gamma\, e^{i\omega t})$.

Solution (a). The natural frequency is

$$\omega_o = \sqrt{169} = 13 \quad 1/\mathrm{sec}\,.$$

The natural period is then

$$T_o = \frac{2\pi}{\omega_o} = \frac{2\pi}{\sqrt{169}} = \frac{2\pi}{13} \quad \mathrm{sec}\,.$$

Remark. You did not need to evaluate $\sqrt{169} = 13$ for full credit.

Solution (b). The characteristic polynomial of the equation is

$$\begin{aligned} p(z) = z^2 + 10z + 169 &= (z+5)^2 + 169 - 25 \\ &= (z+5)^2 + 144 = (z+5)^2 + 12^2\,. \end{aligned}$$

This has the conjugate pair of roots $-5 \pm i12$. Therefore the system is *under damped*. Its damped frequency ω_η is

$$\omega_\eta = \sqrt{144} = 12 \quad 1/\mathrm{sec}\,.$$

The damped period T_η is then

$$T_\eta = \frac{2\pi}{\omega_\eta} = \frac{2\pi}{\sqrt{144}} = \frac{2\pi}{12} = \frac{\pi}{6} \quad \mathrm{sec}\,.$$

Remark. You did not need to evaluate $\sqrt{144} = 12$ for full credit.

Alternative Solution (b). The system is *under damped* because the damping rate $\eta = 5$ is less that the natural frequency $\omega_o = \sqrt{169} = 13$. The damped frequency ω_η is then given by

$$\omega_\eta = \sqrt{\omega_o^2 - \eta^2} = \sqrt{169 - 25} = \sqrt{144} = 12 \quad 1/\mathrm{sec}\,.$$

The damped period T_η is found as before.

Solution (c). The forcing $f(t) = \alpha\cos(\omega t) + \beta\sin(\omega t)$ has the phasor form

$$f(t) = \mathrm{Re}\big(\gamma\, e^{i\omega t}\big)\,, \qquad \text{where the phasor is} \quad \gamma = \alpha - i\beta\,.$$

Therefore the steady state solution has the phasor form

$$h_P(t) = \mathrm{Re}\big(\Gamma\, e^{i\omega t}\big)\,, \qquad \text{where the phasor is} \quad \Gamma = \frac{\gamma}{p(i\omega)}\,.$$

Because $\gamma = \alpha - i\beta$ and $p(z) = z^2 + 10z + 169$, the phasor Γ is

$$\Gamma = \frac{\alpha - i\beta}{169 - \omega^2 + i10\omega}\,.$$

We are not asked to give the solution in either its Cartesian or polar phasor form, so we can stop here.

(12) [8] When a 10 gram mass is hung vertically from a spring, at rest it stretches the spring 20 cm. (Gravitational acceleration is $g = 980$ cm/sec^2.) A dashpot imparts a damping force of 420 dynes (1 dyne = 1 gram cm/sec^2) when the speed of the mass is 3 cm/sec. Assume that the spring force is proportional to displacement, that the damping force is proportional to velocity, and that there are no other forces. At $t = 0$ the mass is displaced 5 cm above its rest position and is released with a downward velocity of 2 cm/sec.

(a) [6] Give an initial-value problem that governs the displacement $h(t)$ for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

(b) [2] Is this system undamped, under damped, critically damped, or over damped? (Give your reasoning!)

Solution (a). Let $h(t)$ be the displacement in centimeters at time t in seconds of the mass from its rest position, with upward displacements being positive. Because there is no external forcing, the governing initial-value problem has the form

$$m\ddot{h} + c\dot{h} + kh = 0\,, \qquad h(0) = 5\,, \quad \dot{h}(0) = -2\,,$$

where m is the mass, c is the damping coefficient, and k is the spring constant. The problem says that $m = 10$ grams. The damping coefficient c is found by equating the damping force imparted by the dashpot when the speed of the mass is 3 cm/sec, which is $c\,3$ dynes, with the force of 420 dynes. This gives $c\,3 = 420$, or

$$c = \frac{420}{3} = 140 \quad \text{dynes sec/cm}\,.$$

The spring constant k is found by equating the force of the spring when it is stetched 20 cm, which is $k\,20$ dynes, with the weight of the mass, which is $mg = 10 \cdot 980$ dynes. This gives $k\,20 = 10 \cdot 980$, or

$$k = \frac{10 \cdot 980}{20} = 490 \quad \text{dynes/cm}\,.$$

Therefore the governing initial-value problem is

$$10\ddot{h} + 140\,\dot{h} + 490h = 0\,, \qquad h(0) = 5\,, \quad \dot{h}(0) = -2\,.$$

Remark. With the equation in normal form the answer is

$$\ddot{h} + 14\,\dot{h} + 49h = 0\,, \qquad h(0) = 5\,, \quad \dot{h}(0) = -2\,.$$

Remark. If we had chosen downward displacements to be positive then the governing initial-value problem would be the same except for the initial conditions, which would be $h(0) = -5$ and $\dot{h}(0) = 2$.

Solution (b). The damping rate is $\eta = 14/2 = 7$. Because $\eta^2 = 49 = \omega_o^2$, the system is *critically damped*.

Alternative Solution (b). The characteristic polynomial is

$$p(z) = z^2 + 14z + 49 = (z + 7)^2\,.$$

This polynomial has the double negative root -7, so the system is *critically damped*.