First In-Class Exam Solutions
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(1) [10] Find an explicit solution for the following initial-value problems and give its interval of definition.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2(2x - x^2)\,, \qquad x(0) = 1\,.$$

Solution. This is a *nonautonomous, separable* equation. The factors $3t^2$ and $2x - x^2$ are continuous everywhere. Because $2x - x^2 = x(2-x)$, its only stationary points are $x = 0$ and $x = 2$. Because $2x - x^2$ is differentiable at these stationary points, no other solution can touch them. Because its initial value 1 lies between the two stationary points 0 and 2, the solution $x(t)$ of the initial-value problem will lie between them for so long as it exists. In other words, the solution $x(t)$ must satisfy $0 < x(t) < 2$ for every t in its interval of definition. But because $3t^2$ is continuous everywhere, the interval of definition must be $(-\infty, \infty)$.

The differential equation has the separated differential form

$$\frac{1}{2x - x^2}\,\mathrm{d}x = 3t^2\,\mathrm{d}t\,,$$

whereby

$$\int \frac{1}{2x - x^2}\,\mathrm{d}x = \int 3t^2\,\mathrm{d}t = t^3 + c_1\,.$$

By the residual (cover up) method we have the partial fraction identity

$$\frac{1}{2x - x^2} = \frac{1}{x(2-x)} = \frac{\frac{1}{2}}{x} + \frac{\frac{1}{2}}{2-x}\,.$$

This identity plus the fact that $|x| = x$ and $|x - 2| = 2 - x$ when $0 < x < 2$ yield

$$\begin{aligned}\int \frac{1}{2x - x^2}\,\mathrm{d}x &= \tfrac{1}{2}\int \frac{1}{x}\,\mathrm{d}x - \tfrac{1}{2}\int \frac{1}{x-2}\,\mathrm{d}x \\ &= \tfrac{1}{2}\log(|x|) - \tfrac{1}{2}\log(|x-2|) + c_2 \\ &= \tfrac{1}{2}\log(x) - \tfrac{1}{2}\log(2-x) + c_2 = \tfrac{1}{2}\log\left(\frac{x}{2-x}\right) + c_2\,.\end{aligned}$$

By setting $c = c_1 - c_2$ we obtain the implicit general solution

$$\tfrac{1}{2}\log\left(\frac{x}{2-x}\right) = t^3 + c\,.$$

The initial condition $x(0) = 1$ implies that

$$\tfrac{1}{2}\log\left(\frac{1}{2-1}\right) = 0^3 + c\,,$$

whereby $c = 0$. Hence, the solution is governed implicitly by

$$\log\left(\frac{x}{2-x}\right) = 2t^3\,.$$

Upon exponentiating both sides we obtain

$$\frac{x}{2-x} = e^{2t^3},$$

which upon multiplying by $2-x$ becomes the linear expression in x given by

$$x = e^{2t^3}(2-x).$$

This can be solved to arrive at the explicit solution

$$x = \frac{2e^{2t^3}}{1+e^{2t^3}}.$$

This foumula confirms that its *interval of definition* is $(-\infty, \infty)$.

(2) [10] Find an explicit solution for the following initial-value problems and give its interval of definition.

$$(z^2-4)\frac{\mathrm{d}y}{\mathrm{d}z} + 6zy = \frac{3}{z^2-4}, \qquad y(0) = 3.$$

Solution. This is a *nonhomogeneous linear* equation. Its normal form is

$$\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{6z}{z^2-4}\,y = \frac{3}{(z^2-4)^2}.$$

Its coefficient $6z/(z^2-4)$ and forcing $3/(z^2-4)^2$ both are undefined at $z = \pm 2$ and are continuous elsewhere. Therefore, because the the initial time is $z = 0$, the *interval of definition* of the solution is $(-2, 2)$.

An integrating factor is

$$\exp\left(\int \frac{6z}{z^2-4}\,\mathrm{d}z\right) = \exp\big(3\log(|z^2-4|)\big) = |z^2-4|^3.$$

Because the interval of definition is $(-2, 2)$ we have

$$|z^2-4|^3 = (4-z^2)^3,$$

whereby the equation has the integrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}z}\big((4-z^2)^3 y\big) = (4-z^2)^3\frac{3}{(z^2-4)^2} = 3(4-z^2) = 12 - 3z^2.$$

Upon integrating both sides of this equation we see that

$$(4-z^2)^3 y = \int 12 - 3z^2\,\mathrm{d}z = 12z - z^3 + c.$$

The initial condition $y(0) = 3$ implies that $(4-0^2)^3 3 = 12\cdot 0 + 0^3 + c$, whereby $c = 4^3\cdot 3 = 64\cdot 3 = 192$. Therefore the solution is

$$y = \frac{12z - z^3 + 192}{(4-z^2)^3}.$$

This foumula confirms that its *interval of definition* is $(-2, 2)$.

(3) [5] Sketch the graph that would be produced by the following Matlab commands.

```
[X, Y] = meshgrid(−4:0.1:4,−4:0.1:4)
contour(X, Y, X.*Y, [−8, −4, −2])
axis square
```

Solution. The meshgrid command says the sketch should show both x and y axes marked from -4 to 4. The contour command plots the graph of the curve $xy = c$ for the values $c = -8$, $c = -4$, and $c = -2$, which are the graphs of the three hyperbolas

$$xy = -8\,, \qquad xy = -4\,, \qquad xy = -2\,.$$

These live in the second and fourth quadrants. A sketch will be shown in discussion.

(4) [5] Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{\cos(t)}{t^2 - 25}\,w = \frac{e^t}{\sin(t)}\,, \qquad w(-4) = 5\,.$$

(Do not solve the equation to answer this question, but give reasoning!)

Solution. This problem is linear in w. It is already in normal form. The interval of definition can be read off as follows.

– First, notice that the coefficient $\cos(t)/(t^2 - 25)$ is undefined at $t = \pm 5$ and is continuous elsewhere.
– Next, notice that the forcing $e^t/\sin(t)$ is undefined at $t = n\pi$ for every integer n and is continuous elsewhere.

Therefore the *interval of definition is* $(-5, -\pi)$ because

- the initial time $t = -4$ is in $(-5, -\pi)$,
- both the coefficient and forcing are continuous over $(-5, -\pi)$,
- the coefficient is undefined at $t = -5$,
- the forcing is undefined at $t = -\pi$.

(5) [10] Consider the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{(u^2 - 4)(u + 6)^2}{(u^2 + 4)(u - 6)}\,.$$

(a) [7] Sketch its phase-line portrait over the interval $-8 \le u \le 8$. Identify points where solutions are undefined with a $\circ$. Identify stationary points with a $\bullet$ and classify each as being either stable, unstable, or semistable.

(b) [3] For each stationary point identify the set of initial-values $u(0)$ such that the solution $u(t)$ converges to that stationary point as $t \to -\infty$.

Solution (a). This equation is autonomous. Its right-hand side is undefined at $u = 6$ and differentiable elsewhere. Its stationary points are found by setting

$$\frac{(u^2 - 4)(u + 6)^2}{(u^2 + 4)(u - 6)} = 0\,.$$

Because $u^2 - 4 = (u+2)(u-2)$, the stationary points are $u = -2$, $u = 2$, and $u = -6$. (Notice that $u^2 + 4 > 0$.) Because the right-hand side is differentiable at each of these stationary points, no other solutions will touch them. (Uniqueness!)

A sign analysis of the right-hand side shows that the phase-line portrait is

$$
\begin{array}{ccccccccccccc}
 & - & & - & & & + & & - & & + & & \\
\leftarrow\leftarrow & + & \leftarrow\leftarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \rightarrow\rightarrow\rightarrow\rightarrow & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \circ & \rightarrow\rightarrow & + & \rightarrow\rightarrow \; u \\
 & -8 & & -6 & & -2 & & 2 & & 6 & & 8 & \\
 & & & \text{semistable} & & \text{unstable} & & \text{stable} & & \text{undefined} & & &
\end{array}
$$

Remark. The terms stable, unstable, and semistable are applied only to solutions. The point $u = 6$ is not a solution, so these terms should not be applied to it.

Solution (b). As t decreases the solutions $u(t)$ will move *opposite the direction of the arrows* shown in the phase-line portrait given in the solution to part (a). Moreover, uniqueness implies that a nonstationary solution will not touch any stationary one.

- The phase-line portrait shows that for the semistable stationary point -6 we have $u(t) \to -6$ as $t \to -\infty$ if and only if $u(0)$ is in the interval $(-\infty, -6]$.
- The phase-line portrait shows that for the unstable stationary point -2 we have $u(t) \to -2$ as $t \to -\infty$ if and only if $u(0)$ is in the interval $(-6, 2)$.
- The phase-line portrait shows that for the stable stationary point 2 we have $u(t) \to 2$ as $t \to -\infty$ if and only if $u(0) = 2$.

(6) [10] Determine if the following differential form is exact. If it is then find an implicit general solution. Otherwise find an integrating factor. (You do not need to find a general solution in the last case.)

$$\big(\cos(x) - \sin(x+y)\big)\, dx + \big(e^y - \sin(x+y)\big)\, dy = 0\,.$$

Solution. This differential form is *exact* because

$$\partial_y\big(\cos(x) - \sin(x+y)\big) = -\cos(x+y) \quad = \quad \partial_x\big(e^y - \sin(x+y)\big) = -\cos(x+y)\,.$$

Therefore we can find $H(x,y)$ such that

$$\partial_x H(x,y) = \cos(x) - \sin(x+y)\,, \qquad \partial_y H(x,y) = e^y - \sin(x+y)\,.$$

Integrating the second equation with respect to y shows

$$H(x,y) = e^y + \cos(x+y) + h(x)\,,$$

which implies that

$$\partial_x H(x,y) = -\sin(x+y) + h'(x)\,.$$

Plugging this expression for $\partial_x H(x,y)$ into the first equation gives

$$-\sin(x+y) + h'(x) = \cos(x) - \sin(x+y)\,,$$

which shows $h'(x) = \cos(x)$. Taking $h(x) = \sin(x)$, an implicit general solution is

$$e^y + \cos(x+y) + \sin(x) = c\,.$$

(7) [10] Consider the following MATLAB function m-file.

```
function [t,x] = solveit(tI, xI, tF, n)

t = zeros(n + 1, 1); x = zeros(n + 1, 1);
t(1) = tI; x(1) = xI; h = (tF - tI)/n; hhalf = h/2;
for k = 1:n
thalf = t(k) + hhalf; t(k + 1) = t(k) + h;
fnow = (t(k))^2 + exp(t(k)*x(k)); xhalf = x(k) + hhalf*fnow;
fhalf = (thalf)^2 + exp(thalf*xhalf); x(k + 1) = x(k) + h*fhalf;
end
```

Suppose the input values are tI = 1, xI = 0, tF = 11, and n = 50.

(a) [4] What initial-value problem is being approximated numerically?
(b) [1] What is the numerical method being used?
(c) [1] What is the step size?
(d) [4] What will be the output values of t(2) and x(2)?

Solution (a). The initial-value problem being approximated numerically is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t^2 + \exp(tx)\,, \qquad x(1) = 0\,.$$

Remark. An initial-value problem consists of both a differential equation and an initial condition. Both must be given for full credit.

Solution (b). The solution is being approximated by the Runge-midpoint method. (This is clear from the "h*fhalf" in last line of the "for" loop.)

Solution (c). Because tF = 11, tI = 1, and n = 50, the step size is

$$\mathrm{h} = \frac{\mathrm{tF} - \mathrm{tI}}{\mathrm{n}} = \frac{11 - 1}{50} = \frac{10}{50} = \frac{1}{5} = 0.2\,.$$

Remark. The correct values for tF, tI, and n had to be plugged in to get full credit.

Solution (d). Because h = 0.2, we have hhalf = 0.1.
Because tI = 1 and xI = 0, we have t(1) = tI = 1, and x(1) = xI = 0.
Setting k = 1 inside the "for" loop then yields

$$\mathrm{thalf} = \mathrm{t(1)\ +\ hhalf} = 1 + 0.1 = 1.1\,,$$

$$\mathrm{t(2)\ =\ t(1)\ +\ h} = 1 + 0.2 = 1.2\,,$$

$$\mathrm{fnow\ =\ (t(1))\hat{\ }2\ +\ exp(t(1)*x(1))} = 1^2 + \exp(1 \cdot 0) = 1 + 1 = 2\,,$$

$$\mathrm{xhalf\ =\ x(1)\ +\ hhalf*fnow} = 0 + 0.1 \cdot 2 = 0.2\,,$$

$$\mathrm{fhalf\ =\ (thalf)\hat{\ }2\ +\ exp(thalf*xhalf)} = (1.1)^2 + \exp(1.1 \cdot 0.2) = (1.1)^2 + \exp(0.22)\,,$$

$$\mathrm{x(2)\ =\ x(1)\ +\ h*fhalf} = 0 + 0.2\left((1.1)^2 + \exp(0.22)\right).$$

Remark. This expression for x(2) did not have to be simplified to get full credit.

(8) [10] Determine if the following differential form is exact. If it is then find an implicit general solution. Otherwise find an integrating factor. (You do not need to find a general solution in the last case.)

$$x^2\,\mathrm{d}x + (x^3 + y^6 + 2y^5)\,\mathrm{d}y = 0\,.$$

Solution. This differential form is *not exact* because

$$\partial_y(x^2) = 0 \qquad \neq \qquad \partial_x(x^3 + y^6 + 2y^5) = 3x^2\,.$$

We seek an integrating factor ρ that satisfies

$$\partial_y\big[x^2\rho\big] = \partial_x\big[(x^3 + y^6 + 2y^5)\rho\big]\,.$$

Expanding the partial derivatives yields

$$x^2\partial_y\rho = (x^3 + y^6 + 2y^5)\partial_x\rho + 3x^2\rho\,.$$

If we set $\partial_x\rho = 0$ then this reduces to $\partial_y\rho = 3\rho$, which yields the integrating factor

$$\rho = e^{3y}\,.$$

Remark. Because the differential form was not exact, all we were asked to do was find an integrating factor. If we had been asked to find an implicit general solution then we would seek $H(x,y)$ such that

$$\partial_x H(x,y) = x^2e^{3y}\,, \qquad \partial_y H(x,y) = (x^3 + y^6 + 2y^5)e^{3y}\,.$$

These equations can be integrated to find $H(x,y) = \frac{1}{3}x^3e^{3y} + \frac{1}{3}y^6e^{3y}$. Therefore an implicit general solution is

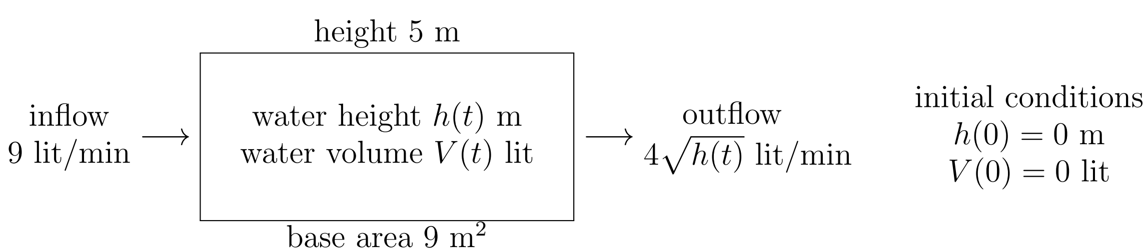
$$x^3e^{3y} + y^6e^{3y} = c\,.$$

(9) [5] A tank has a square base with 3 meter edges, a height of 5 meters, and an open top. It is initially empty when water begins to fill it at a rate of 9 liters per minute. The water also drains from the tank through a hole in its bottom at a rate of $4\sqrt{h}$ liters per minute where $h(t)$ is the height of the water in the tank in meters.

(a) [4] Give an initial-value problem that governs $h(t)$. (Recall 1 m^3 = 1000 lit.) (Do not solve the initial-value problem!)

(b) [1] Does the tank overflow? (Why or why not?)

Solution (a). Let $V(t)$ be the volume (lit) of water in the tank at time t minutes. We have the following (optional) picture.



We want to write down an initial-value problem that governs $h(t)$.

Because the tank has a base with an area of 9 m^2, the volume of water in the tank is $9h(t)$ m^3. Because 1 m^3 = 1000 lit, $V(t) = 1000 \cdot 9h(t) = 9000h(t)$ lit. Because

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{RATE IN} - \text{RATE OUT} = 9 - 4\sqrt{h}\,,$$

and $V = 9000h$, the initial-value problem that governs $h(t)$ is

$$9000\,\frac{\mathrm{d}h}{\mathrm{d}t} = 9 - 4\sqrt{h}\,, \qquad h(0) = 0\,.$$

Each term in the differential equation has units of lit/min.

Solution (b). The differential equation is autonomous. Its right-hand side is defined for $h \geq 0$ and is differentiable for $h > 0$. It has one stationary point at $h = \frac{81}{16}$. Its phase-line portrait for $h > 0$ is

$$\begin{array}{c} \quad + \qquad\qquad\qquad - \\ \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \bullet \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\ h \\ \tfrac{81}{16} \end{array}$$

This portrait shows that if $h(0) = 0$ then $h(t) \to \frac{81}{16}$ as $t \to \infty$. Because the height of the tank is 5 and $\frac{81}{16} > 5$, the tank will overflow.

(10) [7] In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population such that it would triple every seven weeks. There are 320,000 mosquitoes in the area when a flock of birds arrives that eats 50,000 mosquitoes per week.

(a) [5] Give an initial-value problem that governs $M(t)$, the number of mosquitoes in the area after the flock of birds arrives. (Do not solve the initial-value problem!)

(b) [2] Is the flock large enough to control the mosquitoes? (Why or why not?)

Solution (a). The population tripling every seven weeks means that the growth rate r satisfies $e^{r7} = 3$, whereby

$$r = \tfrac{1}{7}\log(3) \quad 1/\text{week}\,.$$

Therefore the initial-value problem that M satisfies is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \tfrac{1}{7}\log(3)M - 50,000\,, \qquad M(0) = 320,000\,.$$

Each term in the differential equation has units of mosquitoes/week.

Solution (b). Because the differential equation is autonomous (as well as linear), the monotonicity of $M(t)$ can be determined by a sign analysis of its right-hand side. We see from part (a) that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \tfrac{1}{7}\log(3)\left(M - \frac{350,000}{\log(3)}\right) \text{ is } \begin{cases} < 0 & \text{for } M < \dfrac{350,000}{\log(3)}\,, \\ > 0 & \text{for } M > \dfrac{350,000}{\log(3)}\,. \end{cases}$$

This can be visualized with the phase-line portrait, which is

$$\begin{array}{c} \quad - \qquad\qquad\qquad + \\ \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow \bullet \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\ M \\ \tfrac{350{,}000}{\log(3)} \end{array}$$

Therefore the flock is large enough to control the mosquitoes if

$$M(0) = 320,000 < \frac{350,000}{\log(3)},$$

and it is not large enough to control the mosquitoes if

$$M(0) = 320,000 > \frac{350,000}{\log(3)}.$$

You got full marks for this answer. Because $\log(3) > \frac{35}{32}$, the flock is not large enough to control the mosquitoes.

Remark. The problem was supposed to say that 350,000 mosquitoes are in the area when a flock of birds arrives, which makes the answer somewhat more straightforward. I am sorry for this typo.

(11) [10] A puck with initial velocity $v_o > 0$ begins to slide on a surface that imparts a position-dependent frictional drag. Its position $x(t)$ is governed by the initial-value problem

$$\ddot{x} = -e^{-x}\dot{x}, \qquad x(0) = 0, \quad \dot{x}(0) = v_o > 0.$$

(a) [8] Solve the auxiliary equation and write down the resulting reduced equation.
(b) [2] Find the smallest initial velocity v_o for which $x(t) \to \infty$ as $t \to \infty$.

Solution (a). This is a second-order autonomous initial-value problem. Its auxiliary equation is

$$v\,\frac{\mathrm{d}v}{\mathrm{d}x} = -e^{-x}v.$$

This is satisfied if and only if either

$$v = 0, \qquad \text{or} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = -e^{-x}.$$

Because $v_o > 0$ we can eliminate the first case. The second case is an explicit differential equation for v. Its general solution is

$$v = e^{-x} + c.$$

The initial conditions imply that $v = v_o$ when $x = 0$, whereby

$$v_o = e^{-0} + c = 1 + c.$$

Hence, $c = v_o - 1$ and the solution of the auxiliary equation is

$$v = v_o - 1 + e^{-x}.$$

Therefore the resulting reduced equation is

$$\dot{x} = v_o - 1 + e^{-x}.$$

Solution (b). The reduced equation is the first-order autonomous equation

$$\dot{x} = v_o - 1 + e^{-x} .$$

Its stationary solutions must satisfy

$$0 = v_o - 1 + e^{-x} .$$

This has a solution if and only if $v_o < 1$, in which case the only stationary point is

$$x = x_\infty = \log\left(\frac{1}{1 - v_o}\right) .$$

Because $v_o > 0$ we see that $x_\infty > 0$. Therefore we have two cases to consider.

- When $0 < v_o < 1$ the reduced equation has the single stationary point x_∞ and its phase-line portrait is

$$\begin{array}{c} \quad + \qquad\qquad\qquad - \\ \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \bullet \leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow \; x \\ x_\infty \end{array}$$

  Because $x(0) = 0 < x_\infty$, we see that $\dot{x}(t) > 0$ and that $x(t) \to x_\infty$ as $t \to \infty$. Therefore we do not have $x(t) \to \infty$ as $t \to \infty$ when $0 < v_o < 1$.

- When $v_o \geq 1$ the reduced equation has no stationary points and its phase-line portrait is

$$\begin{array}{c} + \\ \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow \; x \end{array}$$

  We see that $\dot{x}(t) > 0$ and that $x(t) \to \infty$ as $t \to \infty$. Therefore we have $x(t) \to \infty$ as $t \to \infty$ when $v_o \geq 1$.

Therefore $v_o = 1$ is the smallest initial velocity for which $x(t) \to \infty$ as $t \to \infty$.

Remark. Full marks required arguing (1) that $x(t) \to \infty$ as $t \to \infty$ when $v_o = 1$ and (2) that this does not happen when $0 < v_o < 1$.

Remark. The reduced equation may be solved explicitly, but that is not the best way to approach the problem. The explicit solution is

$$x(t) = \begin{cases} \log\left(\dfrac{1 - v_o e^{-(1-v_o)t}}{1 - v_o}\right) & \text{for } v_o < 1 \,, \\ \log(1+t) & \text{for } v_o = 1 \,, \\ \log\left(\dfrac{v_o e^{(v_o-1)t} - 1}{v_o - 1}\right) & \text{for } v_o > 1 \,. \end{cases}$$

This shows that $v_o = 1$ is the smallest initial velocity for which $x(t) \to \infty$ as $t \to \infty$. To obtain this solution we had to first find an implicit general solution by integrating

$$\int \frac{1}{v_o - 1 + e^{-x}} \, \mathrm{d}x = t + c \,.$$

The integral on the left-hand side is similar to one from the sample problems.

(12) [8] Suppose you have used a numerical method to approximate the solution of an initial-value problem over the time interval $[2, 10]$ with 800 uniform time steps. What step size is needed to reduce the global error of your approximation by a factor of $\frac{1}{256}$ if the method you had used was each of the following? (Notice that $256 = 4^4$.)

(a) explicit Euler method
(b) Runge-Kutta method
(c) Runge-midpoint method
(d) Runge-trapezoidal method

Remark. Notice that the step size used in the original calculation is

$$h = \frac{t_F - t_I}{N} = \frac{10 - 2}{800} = \frac{1}{100}\,.$$

Solution (a). The explicit Euler method is first order, so its error scales like h. To reduce the error by a factor of $\frac{1}{256}$, we must reduce h by a factor of $\frac{1}{256}$. Because the original h was $\frac{1}{100}$, we must set

$$h = \frac{1}{100} \cdot \frac{1}{256} = \frac{1}{25600}\,.$$

Solution (b). The Runge-Kutta method is fourth order, so its error scales like h^4. To reduce the error by a factor of $\frac{1}{256}$, we must reduce h by a factor of $\frac{1}{256}^{\frac{1}{4}} = \frac{1}{4}$. Because the original h was $\frac{1}{100}$, we must set

$$h = \frac{1}{100} \cdot \frac{1}{4} = \frac{1}{400}\,.$$

Solution (c). The Runge-midpoint method is second order, so its error scales like h^2. To reduce the error by a factor of $\frac{1}{256}$, we must reduce h by a factor of $\frac{1}{256}^{\frac{1}{2}} = \frac{1}{16}$. Because the original h was $\frac{1}{100}$, we must set

$$h = \frac{1}{100} \cdot \frac{1}{16} = \frac{1}{1600}\,.$$

Solution (d). The Runge-trapezoidal method is second order, so its error scales like h^2. To reduce the error by a factor of $\frac{1}{256}$, we must reduce h by a factor of $\frac{1}{256}^{\frac{1}{2}} = \frac{1}{16}$. Because the original h was $\frac{1}{100}$, we must set

$$h = \frac{1}{100} \cdot \frac{1}{16} = \frac{1}{1600}\,.$$

Remark. The number of time steps needed to reduce the error by a factor of $\frac{1}{256}$ is respectively

$$\text{(a) } 800 \cdot 256\,, \qquad \text{(b) } 800 \cdot 4\,, \qquad \text{(c) } 800 \cdot 16\,, \qquad \text{(d) } 800 \cdot 16\,.$$

Had you computed these then the associated step sizes can be expressed as

$$\text{(a) } \frac{10-2}{800 \cdot 256}\,, \qquad \text{(b) } \frac{10-2}{800 \cdot 4}\,, \qquad \text{(c) } \frac{10-2}{800 \cdot 16}\,, \qquad \text{(d) } \frac{10-2}{800 \cdot 16}\,.$$

The arithmetic did not have to be carried out for full credit.