Introduction to Linear Semigroup Theory

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1. Introduction

A fundamental question about partial differential equations is when are they well-posed. This question has its roots in the very beginning of the subject. In 1746 Jean le Rond d’Alembert derived the wave equation

$$\partial_{tt}u - \partial_{xx}u = 0,$$

(1)

to describe the motion of a vibrating string. He discovered that its solution over the line can be expressed as

$$u(t, x) = F(x - t) + G(x + t),$$

(2)

where $F$ and $G$ are determined by the initial data. In 1748 Leonhard Euler extended this method to study the plucked string. However, d’Alembert did not approve of this extension because the resulting $F$ and $G$ were not differentiable, and therefore were not solutions of his wave equation!
In 1753 Daniel Bernoulli introduced the method of separation of variables to solve the wave equation. He gave formulas for the waves that had been described by Brook Taylor in 1714. He asserted that general solutions could be written as the sum of such solutions — i.e. as a Fourier series.

In 1807 Joseph Fourier proposed that the dynamics of caloric (the massless substance that was then thought to carry heat) was governed by the heat equation

$$\partial_t u = \partial_{xx} u. \quad (3)$$

He solved this by separation of variables and expressed solutions as a Fourier series. Fourier went far beyond his predecessors by asserting that this method applies to initial data with jump discontinuities. This assertion was so controversial that his article did not appear until 1822.
In 1827 Pierre-Simon Laplace expressed the solution of the heat equation for $t > 0$ as

$$u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x - y)^2}{4t}\right) u_o(x) \, dy,$$

where $u_o$ is the initial data. This formula clearly applies to any initial data that is integrable. Of course, what it means to be integrable was not fully understood.

Bernhard Riemann developed his theory of the integral in 1854, but it was not published in a journal until 1868 when it appeared as an appendix in a paper on Fourier series. The theory of Gaston Darboux appeared in 1875. The extension of their theories by Thomas Joannas Stieltjes appeared in 1894. The theory of Henri Lebesgue appeared in his 1902 thesis and was published in a journal in 1904.
In 1902 Jacques Hadamard introduced the notion of a *well-posed* problem. He said that a mathematical problem was *well-posed* if

- its solution exists;
- its solution is unique;
- its solution depends continuously on parameters.

Here parameters is a notion that includes initial data, boundary data, and coefficients in the equation. Ideally, the solution depends continuously on a broad class of perturbations of the equation itself. Hadamard argued that these properties where crucial if a mathematical problem was to be useful in science and engineering.
In 1948 E. Hille and K. Yosida independently characterized certain classes of well-posed initial-value problems in Banach spaces. More specifically, for a given Banach space $X$ it characterizes those operators $A$ such that for every $u_o \in X$ the initial-value problem

$$\partial_t u = Au, \quad u(0) = u_o, \quad (5)$$

is well-posed in a precise sense that we will define later.

The heat equation (3) is already in the form (5) with $A = \partial_{xx}$. Two ways to put the wave equation (1) into the form (5) are

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ \partial_{xx} & 0 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}.$$ 

There are other ways to do it!
Of course, if $A \in \mathcal{B}(X)$ then (5) is well-posed. Its solution is given by

$$u(t) = S(t)u_o,$$

where $S(t) = \exp(tA) \in \mathcal{B}(X)$ is the unique solution in $C(\mathbb{R}; \mathcal{B}(X))$ of the initial-value problem

$$\partial_t S = AS, \quad S(0) = I,$$

(6)

The existence and uniqueness of $S(t)$ is classical. We can show that

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

$$= \lim_{n \to \infty} \left( I + \frac{t}{n} A \right)^n$$

$$= \lim_{n \to \infty} \left( I - \frac{t}{n} A \right)^{-n}.$$

The second equality asserts the convergence of the explicit Euler method. The third equality asserts the convergence of the implicit Euler method.
The operator $S(t) = \exp(tA)$ has the properties

(i) $S(s + t) = S(s)S(t)$ for every $s, t \in \mathbb{R}$;
(ii) $S(0) = I$;
(iii) $\lim_{s \to t} S(s) = S(t)$ for every $t \in \mathbb{R}$. \hfill (7)

Conversely, if $S(t)$ has properties (7) then there exists $A \in \mathcal{B}(X)$ such that $S(t) = \exp(tA)$. (See *Functional Analysis* by Lax.)

Therefore this theory cannot be apply to PDEs because in that case the operator $A$ is unbounded.
2. Strongly Continuous One-Parameter Semigroups

We assert that solutions of the intial-value problem (5) is well-posed if $u(t) = S(t)u_0$ for every $t \in [0, \infty)$ where $S(t) \in \mathcal{B}(X)$ satisfies

(i) $S(s + t) = S(s)S(t)$ for every $s, t \in [0, \infty)$;

(ii) $S(0) = I$;

(iii) $\lim_{t \to 0^+} S(t)v = v$ for every $v \in X$.

These differ from the properties (7) in two ways.

- $S(t)$ is defined only for $t \in [0, \infty)$ rather than for $t \in \mathbb{R}$.

- $S(t)v$ is only strongly continuous at 0 rather than $S(t)$ is uniformly continuous in $t$. 
We can show that if $S(t) \in B(X)$ satisfies (8) then

- there exists $a \in \mathbb{R}$ and $b \geq 1$ such that
  \[ \|S(t)\| \leq be^{at}; \]  
  (9a)

- For every $v \in X$ and $t \in (0, \infty)$ we have
  \[ \lim_{s \to t} S(s)v = S(t)v. \]  
  (9b)

The first item is a consequence of the uniform boundedness principle. The second states that $S(t)v$ is a strongly continuous function of $t$ over $[0, \infty)$. Therefore if $S(t)$ satisfies (8) we call it a strongly continuous one-parameter semigroup.
Remark. In property (iii) of (8) we can replace strong convergence with weaker notions of convergence. For example, we can assume that

\[ \lim_{t \to 0^+} w^{-} S(t)v = v, \]

or if \( X \) is itself a dual space then we can assume that

\[ \lim_{t \to 0^+} w^{*-} S(t)v = v. \]

The first of these is equivalent to (iii) of (8), however the second is weaker. For example, for \( X = L^\infty(\mathbb{R}) \) the solution of the heat equation given by (4) does not satisfy property (iii) of (8), but does satisfy the second property above.

Remark. We claim that initial-value problem (5) is well-posed when is solution operator \( S(t) \) is a strongly continuous one-parameter semigroup.
Let $S(t)$ be a strongly continuous one-parameter semigroup. Define an operator $A$ by

$$Av = \operatorname{s-lim}_{t \to 0^+} \frac{S(t)v - v}{t},$$

(10a)

for every $v \in \operatorname{Dom}(A)$ where

$$\operatorname{Dom}(A) = \left\{ v \in X : \operatorname{s-lim}_{t \to 0^+} \frac{S(t)v - v}{t} \text{ exists} \right\}.$$  

(10b)

The subspace $\operatorname{Dom}(A)$ is the domain of $A$. The operator $A$ is called the \textit{infinitesimal generator} or simply the \textit{generator} of the strongly continuous one-parameter semigroup $S(t)$. We now collect properties of $A$. 

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**3. Generators of Strongly Continuous Semigroups**
• If \( v \in \text{Dom}(A) \) then \( S(t)v \in \text{Dom}(A) \) and
\[
A S(t)v = S(t)A v \quad \text{for every } t \in [0, \infty).
\]

• \( \text{Dom}(A) \) is a dense linear subspace of \( X \).

• \( \text{Dom}(A^n) \) is a dense linear subspace of \( X \) for every \( n \in \mathbb{Z}_+ \) where
\[
\text{Dom}(A^n) = \left\{ v \in \text{Dom}(A^{n-1}), A^{n-1}v \in \text{Dom}(A) \right\}.
\]

• If \( v \in \text{Dom}(A^n) \) then \( S(t)v \in \text{Dom}(A^n) \) and
\[
A^n S(t)v = S(t)A^n v \quad \text{for every } t \in [0, \infty).
\]
**Proof Sketch.** By semigroup properties (i) and (ii) of (8), for every $v \in X$, $t > 0$, and $h > 0$ we have

$$\frac{S(t + h) - S(t)}{h} v = S(t) \frac{S(h) - I}{h} v = \frac{S(h) - I}{h} S(t) v.$$

If $v \in \text{Dom}(A)$ then the middle term converges to $S(t)Av$, whereby $S(t)v$ is differentiable and $S(t)v \in \text{Dom}(A)$ with

$$\partial_t S(t)v = S(t)Av = AS(t)v.$$
Next, we claim that for every \( v \in X \) we have

\[
\int_0^t S(s)v \, ds \in \text{Dom}(A) \quad \text{and} \quad A \int_0^t S(s)v \, ds = S(t)v - v. \quad (11)
\]

For every \( h > 0 \) we have

\[
\frac{S(h) - I}{h} \int_0^t S(s)v \, ds = \frac{1}{h} \int_0^t (S(s + h)v - S(s)v) \, ds = \int_t^{t+h} S(s)v \, ds - \frac{1}{h} \int_0^h S(s)v \, ds.
\]

Because \( S(s)v \) is continuous we have

\[
\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} S(s)v \, ds = S(t)v, \quad \lim_{h \to 0^+} \frac{1}{h} \int_0^h S(s)v \, ds = v.
\]

Therefore our claim is established.
Because for arbitrary $v \in X$ we have

$$\frac{1}{t} \int_0^t S(s)v \, ds \in \text{Dom}(A) \quad \text{for every } t > 0,$$

and

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t S(s)v \, ds = v,$$

we conclude that $\text{Dom}(A)$ is dense in $X$.

The assertions for $n > 1$ go similarly, but we skip that part of the proof.
Moreover the operator $A$ is closed. This means that the graph of $A$ is a closed set in $X \times X$. This means that if $\{u_n\}$ is a sequence contained in $\text{Dom}(A)$ such that if there exists $(u, v) \in X \times X$ such that

$$s-\lim_{n \to \infty} u_n = u \quad \text{and} \quad s-\lim_{n \to \infty} Au_n = v,$$

then $u \in \text{Dom}(A)$ and $Au = v$. We prove this by passing to the limit in

$$S(t)u_n - u_n = \int_0^t S(s)Au_n \, ds,$$

which leads to

$$S(t)u - u = \int_0^t S(s)v \, ds.$$

Because $S(s)v$ is continuous we have

$$\lim_{t \to 0^+} \frac{S(t)u - u}{t} = \lim_{t \to 0^+} \frac{1}{t} \int_0^t S(s)v \, ds = v.$$

We conclude that $u \in \text{Dom}(A)$ and $Au = v$. Therefore $A$ is closed.
Finally, for all $\zeta \in \mathbb{C}$ such that $\text{Re}(\zeta) > a$ the operator $\zeta I - A$ has a bounded inverse that satisfies the bounds

$$
\left\| (\zeta I - A)^{-k} \right\| \leq \frac{b}{(\text{Re}(\zeta) - a)^k} \quad \text{for every } k \in \mathbb{Z}_+ ,
$$

(12)

where $a \in \mathbb{R}$ and $b \geq 1$ are the constants appearing in the bound (9a).

For every $v \in X$ consider the Laplace transform of $S(t)v$, which is

$$
\mathcal{L}[S(t)v](\zeta) = \int_0^\infty e^{-\zeta t} S(t)v \, dt .
$$

Bound (9a) implies that

$$
\| \mathcal{L}[S(t)v](\zeta) \| \leq \int_0^\infty |e^{-\zeta t}| \| S(t) \| \, dt \| v \|
\leq b \int_0^\infty e^{-\text{Re}(\zeta) t} e^{at} \, dt \| v \| = \frac{b}{\text{Re}(\zeta) - a} \| v \| .
$$
It follows that $L[S(t)v](\zeta)$ defines a bounded linear operator over $X$. Let $R(\zeta) \in B(X)$ be defined by $R(\zeta)v = L[S(t)v](\zeta)$ for every $v \in X$. We claim that for every $\zeta \in \mathbb{C}$ such that $\text{Re}(\zeta) > a$ and every $v \in X$ we have

$$R(\zeta)v \in \text{Dom}(A) \quad \text{and} \quad (\zeta I - A)R(\zeta)v = v.$$ 

It would then follow that $R(\zeta) = (\zeta I - A)^{-1}$, which is the resolvent of $A$.

From (11) we can show that

$$e^{-\zeta t}S(t)v - v = (A - \zeta I) \int_0^t e^{-\zeta s}S(s)v \, ds.$$ 

By letting $t \to \infty$ while using bound (9a), the fact that $\text{Re}(\zeta) > a$, and the fact that $A - \zeta I$ is closed, we obtain

$$-v = (A - \zeta I) \int_0^\infty e^{-\zeta s}S(s)v \, ds = (A - \zeta I)R(\zeta)v.$$ 

Therefore our claim is established. The resolvent bound (12) for $k = 1$ follows.
The resolvent bounds (12) for $k > 1$ follow by first proving the identity

$$(\zeta I - A)^{-k}v = \frac{1}{(k - 1)!} \int_0^\infty t^{k-1} e^{-\zeta t} S(t)v \, dt.$$ 

This identity holds for every $k \in \mathbb{Z}_+$, every $v \in X$, and every $\zeta \in \mathbb{C}$ with $\text{Re}(\zeta) > a$. It follows that

$$\| (\zeta I - A)^{-k}v \| \leq \frac{1}{(k - 1)!} \int_0^\infty t^{k-1} |e^{-\zeta t}| \| S(t) \| \, dt \| v \|$$

$$\leq \frac{b}{(k - 1)!} \int_0^\infty t^{k-1} e^{-\text{Re}(\zeta)t} e^{at} \, dt \| v \|$$

$$= \frac{b}{(\text{Re}(\zeta) - a)^k} \| v \|.$$

Therefore the resolvent bounds (12) hold for every $k \in \mathbb{Z}_+$. 
Remark. For every $B \in \mathcal{B}(X)$ its spectral radius is defined by

$$ \rho_{\text{Sp}}(B) = \sup \{ |\lambda| : \lambda \in \text{Sp}(B) \} ,$$

where $\text{Sp}(B)$ is the spectrum of $B$. It satisfies the Gelfand spectral radius identity

$$ \rho_{\text{Sp}}(B) = \inf \left\{ \| B^k \|^{\frac{1}{k}} : k \in \mathbb{Z}_+ \right\} = \limsup_{k \to \infty} \| B^k \|^{\frac{1}{k}} .$$

From this identity and the resolvent bounds (12) we see that for every $\zeta \in \mathbb{C}$ with $\text{Re}(\zeta) > a$ we have

$$ \rho_{\text{Sp}} \left( (\zeta I - A)^{-1} \right) \leq \frac{1}{\text{Re}(\zeta) - a} .$$

By the spectral mapping theorem every $\lambda \in \text{Sp}(A)$ satisfies

$$ \text{Re}(\zeta) - a \leq |\zeta - \lambda| ,$$

which is equivalent to $\text{Re}(\lambda) \leq a$. In other words, $\text{Sp}(A)$ must lie in the half-plane $\{ \lambda : \text{Re}(\lambda) \leq a \}$. 
4. Hille-Yosida Theorem

If $S(t)$ is a strongly continuous one-parameter semigroup over a Banach space $X$ that satisfies the bound

$$\|S(t)\| \leq be^{at} \quad \text{for some } a \in \mathbb{R} \text{ and } b \geq 1,$$  \hspace{1cm} (13)

then its generator $A$ is a densely defined closed linear operator over $X$ such that for every $\zeta \in \mathbb{C}$ with $\text{Re}(\zeta) > a$ the operator $\zeta I - A$ is invertible and satisfies the resolvent bounds

$$\|(\zeta I - A)^{-k}\| \leq \frac{b}{(\text{Re}(\zeta) - a)^k} \quad \text{for every } k \in \mathbb{Z}_+.$$  \hspace{1cm} (14)

Conversely, if $A$ is a densely defined closed linear operator over $X$ such that for every $\zeta \in \mathbb{C}$ with $\text{Re}(\zeta) > a$ the operator $\zeta I - A$ is invertible and satisfies the resolvent bounds (14) then it is the generator of a strongly continuous one-parameter semigroup over $X$ that satisfies the bound (13).
Remark. We have already established the forward part of the theorem. What remains is to prove the converse part of the theorem. This is the harder direction. Let be $A$ be a densely defined closed linear operator over $X$ such that for every $\zeta \in \mathbb{C}$ with $\text{Re}(\zeta) > a$ the operator $\zeta I - A$ is invertible and satisfies the bounds (14) for some $a \in \mathbb{R}$ and $b \geq 1$. We must construct a strongly continuous one-parameter semigroup $S(t)$ over $X$ that satisfies the bound (13). Hille did this with the construction

$$S(t)v = \lim_{n \to \infty} \left( I - \frac{t}{n} A \right)^{-n} v.$$ 

Yosida did this with the construction

$$S(t)v = \lim_{n \to \infty} \exp(tA_n)v, \quad \text{where } A_n = n(nR(n) - I).$$

Here $R(n) = (nI - A)^{-1}$ for $n > a$. We will take Yosida’s approach.
Remark. We note that strongly continuous one-paramenter semigroups are uniquely determined by their generators. Suppose that $S(t)$ and $T(t)$ are strongly continuous one-paramenter semigroups with the same generator $A$. Then for every $v \in \text{Dom}(A)$ and every $0 \leq s \leq t$ we have

$$\partial_s \left( S(s)T(t - s)v \right) = S(s)AT(t - s)v - S(s)AT(t - s)v = 0.$$ 

Because $S(s)T(t - s)v$ is independent of $s$, we have $S(t)v = T(t)v$. Because $\text{Dom}(A)$ is dense in $X$, we conclude that $S(t) = T(t)$ for every $t \in [0, \infty)$. 
Proof Sketch. For every $n > a$ we have defined

$$A_n = n^2 R(n) - nI,$$

where $R(n) = (nI - A)^{-1}$.

We claim that $A_n$ approximates $A$ in the sense that

$$\lim_{n \to \infty} A_n u = A u \quad \text{for every } u \in \text{Dom}(A). \quad (15)$$

We begin with the identity

$$nR(n) - I = AR(n).$$

Therefore if $v \in \text{Dom}(A)$ we have

$$\lim_{n \to \infty} \|nR(n)v - v\| = \lim_{n \to \infty} \|AR(n)v\| = \lim_{n \to \infty} \|R(n)Av\|$$

$$\leq \lim_{n \to \infty} \|R(n)\| \|Av\| \leq \lim_{n \to \infty} \frac{b}{n - a} \|Av\| = 0.$$
Because $\|nR(n)\|$ is bounded by

$$\|nR(n)\| \leq \frac{bn}{n-a} \quad \text{for every } n > a,$$

and because $\text{Dom}(A)$ is dense in $X$, we have the limit

$$\text{s- lim}_{n \to \infty} nR(n)v = v \quad \text{for every } v \in X. \quad (16)$$

Now we observe that for every $u \in \text{Dom}(A)$ we have

$$A_nu = nAR(n)u = nR(n)Au,$$

whereby the limit (16) implies that

$$\text{s- lim}_{n \to \infty} A_nu = \text{s- lim}_{n \to \infty} nR(n)Au = Au.$$

Therefore we have proved claim (15).
We now consider $S_n(t) = \exp(t A_n)$. We first bound it. For every $t \geq 0$ we have

$$\exp(t A_n) = e^{-nt} \exp(t n^2 R(n)) = e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k} R(n)^k}{k!} t^k.$$ 

Therefore we have the bound

$$\|S_n(t)\| = \|\exp(t A_n)\| \leq e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k} \|R(n)^k\|}{k!} t^k \leq b e^{-nt} \sum_{k=0}^{\infty} \frac{n^{2k} t^k}{k! (n - a)^k} \leq b \exp\left(\left(a + \frac{a^2}{n - a}\right) t\right).$$ 

(17)
Next we show that for every \( u \in \text{Dom}(A) \) and every \( t \geq 0 \) the sequence \( \{S_n(t)u\} \) is Cauchy in \( X \). We begin with the identity for every \( 0 \leq s \leq t \) that

\[
\partial_s\left(S_m(s)S_n(t - s)\right) = S_m(s)S_n(t - s)(A_m - A_n).
\]

Here we have used the fact that all of the operators above commute. By integrating this identity over \( s \) from 0 to \( t \) we obtain

\[
S_m(t) - S_n(t) = \int_0^t S_m(s)S_n(t - s) \, ds \, (A_m - A_n).
\]

Therefore

\[
\|S_m(t)u - S_n(t)u\| \leq \int_0^t \|S_m(s)\| \|S_n(t - s)\| \, ds \|A_mu - A_nu\|.
\]

Because the sequence \( \{A_nu\} \) converges to \( Au \) in \( X \) while the integral can be uniformly bounded by using the bound (17), we conclude that the sequence \( \{S_n(t)u\} \) is Cauchy in \( X \).
Because the sequence \( \{S_n(t)u\} \) is Cauchy in \( X \) for every \( u \in \text{Dom}(A) \), because \( \|S_n(t)\| \) is uniformly bounded by (17), and because \( \text{Dom}(A) \) is dense in \( X \), we conclude the existence of the limit

\[
s\lim_{n \to \infty} S_n(t)v = S(t)v \quad \text{for every } v \in X.
\]

This limit is uniform over bounded sets of \( t \), so \( S(t)v \) is continuous in \( t \). Therefore \( S(t) \) is strongly continuous over \([0, \infty)\).

It is easy to show that \( S(t) \) is a semigroup by passing to the limit in the group properties (7) that are satisfied by \( S_n(t) \). Therefore \( S(t) \) is a strongly continuous one-parameter semigroup.

It is also easy to show that \( \|S(t)\| \) satisfies the bound (13) by passing to the limit in the bound (17).
What remains is to show that $A$ is the generator of $S(t)$. We do this by passing to the limit in

$$S_n(t)u - u = A_n \int_0^t S_n(s)u \, ds.$$  

This yields (details skipped)

$$S(t)u - u = A \int_0^t S(s)u \, ds,$$

Then if $u \in \text{Dom}(A)$ we have

$$\text{s- lim}_{t \to 0^+} \frac{S(t)u - u}{t} = \text{s- lim}_{t \to 0^+} \frac{1}{t} \int_0^t S(s)Au \, ds = Au.$$

We have now proved the Hille-Yosida Theorem.
5. Conclusion

In order to apply the Hille-Yosida Theorem to a linear operator $A$ over a Banach space $X$, we must show that:

- $A$ is densely defined,
- $A$ is closed,
- $A$ satisfies the resolvent bounds (14) for some $a \in \mathbb{R}$ and $b \geq 1$.

In practice, the first item is usually easy to verify. The second and third can be easy to verify, but might not be. Some special cases will illustrate this.
The first special case \((b = 1)\) is the following.

If \(S(t)\) is a strongly continuous one-parameter semigroup over a Banach space \(X\) that satisfies the bound

\[
\|S(t)\| \leq e^{at} \quad \text{for some } a \in \mathbb{R},
\]  

(18)

then its generator \(A\) is a densely defined closed linear operator over \(X\) such that for every \(\zeta \in \mathbb{C}\) with \(\text{Re}(\zeta) > a\) the operator \(\zeta I - A\) is invertible and satisfies the resolvent bound

\[
\left\|(\zeta I - A)^{-1}\right\| \leq \frac{1}{\text{Re}(\zeta) - a}.
\]  

(19)

Conversely, if \(A\) is a densely defined closed linear operator over \(X\) such that for every \(\zeta \in \mathbb{C}\) with \(\text{Re}(\zeta) > a\) the operator \(\zeta I - A\) is invertible and satisfies the resolvent bound (19) then it is the generator of a strongly continuous one-parameter semigroup over \(X\) that satisfies the bound (18).
The contracting semigroup case \((b = 1 \text{ and } a = 0)\) is the following.

If \(S(t)\) is a strongly continuous one-parameter semigroup over a Banach space \(X\) that satisfies the bound

\[
\|S(t)\| \leq 1,
\]  

then its generator \(A\) is a densely defined closed linear operator over \(X\) such that for every \(\zeta \in \mathbb{C}\) with \(\text{Re}(\zeta) > 0\) the operator \(\zeta I - A\) is invertible and satisfies the resolvent bound

\[
\left\| (\zeta I - A)^{-1} \right\| \leq \frac{1}{\text{Re}(\zeta)}.
\]  

Conversely, if \(A\) is a densely defined closed linear operator over \(X\) such that for every \(\zeta \in \mathbb{C}\) with \(\text{Re}(\zeta) > 0\) the operator \(\zeta I - A\) is invertible and satisfies the resolvent bound (21) then it is the generator of a strongly continuous one-parameter semigroup over \(X\) that satisfies the bound (20).
Remark. The resolvent bound (19) can be established by proving that for every \( \zeta \in \mathbb{C} \) with \( \text{Re}(\zeta) > a \) we have

\[
\| (\zeta I - A)v \| \geq (\text{Re}(\zeta) - a) \|v\| \quad \text{for every } v \in \text{Dom}(A),
\]

and that the dual operator \( A' \) satisfies a similar bound. For any closed operator \( A \) such bounds on \( A \) imply that \( \zeta I - A \) is one-to-one and has a closed range while the similar bounds on \( A' \) imply that the range of \( \zeta I - A \) is dense. In such cases \( \zeta I - A \) has a bounded inverse that satisfies the resolvent bound (19).

In particular, the resolvent bound (21) can be established by proving that for every \( \zeta \in \mathbb{C} \) with \( \text{Re}(\zeta) > 0 \) we have

\[
\| (\zeta I - A)v \| \geq \text{Re}(\zeta) \|v\| \quad \text{for every } v \in \text{Dom}(A),
\]

and that the dual operator \( A' \) satisfies a similar bound.
A final special case is the Lumer-Phillips Theorem, which is set in a Hilbert space $X$ with scalar product $(\cdot|\cdot)$.

If $S(t)$ is a strongly continuous one-parameter semigroup over a Hilbert space $X$ that satisfies the bound

$$
\|S(t)\| \leq 1,
$$

(22)

then its generator $A$ is a densely defined closed linear operator over $X$ such that

$$
\text{Re}(v|Av) \leq 0 \quad \text{for every } v \in X.
$$

(23)

Conversely, if $A$ is a densely defined closed linear operator over $X$ such that the operator satisfies the bound (23) then it is the generator of a strongly continuous one-parameter semigroup over $X$ that satisfies the bound (22).
If $A$ generates a strongly continuous one-parameter semigroup on $X$ then so does $A + B$ if

- $BR(n)$ is compact for some $n > a$,

- $BR(n)$ is sufficiently small for some $n > a$.

Here $R(n) = (nI - A)^{-1}$. If $S(t)$ and $T(t)$ are the semigroups generated by $A$ and $A + B$ respectively, then we can bound $\|S(t) - T(t)\|$ in the last case.
Consider the nonhomogeneous linear initial-value problem in the form
\[ \partial_t u = Au + f(t), \quad u(0) = u_o. \]
If \( S(t) \) is the strongly continuous one-parameter semigroup generated by \( A \) then, provided the integral makes sense, this problem has the solution
\[ u(t) = S(t)u_o + \int_0^t S(t - s)f(s)\,ds, \]
Now consider nonlinear initial-value problems in the form
\[ \partial_t u = Au + N(u), \quad u(0) = u_o. \]
This problem has the mild formulation
\[ u(t) = S(t)u_o + \int_0^t S(t - s)N(u(s))\,ds. \]
Fixed point arguments can sometimes be used to show that this has a solition. This often requires more knowledge of the theory of semigroups.