Self-Similar Solutions to the 3D Navier-Stokes Equations

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1 Introduction

We are interested in studying the Cauchy problem for the incompressible 3D Navier-Stokes Equations (NSE), a vector valued PDE written as

$$\partial_t v + (v \cdot \nabla) v = \Delta v - \nabla p, \tag{1.1}$$
$$\nabla \cdot v = 0, \tag{1.2}$$
$$v(0) = v_0, \tag{1.3}$$

where $v(t,x) : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ is thought of as the velocity field of some fluid and $p(t,x) : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ as the scalar pressure.

The first things we notice is that the unknown we are solving for is v, but annoyingly this is not a closed form equation for v. However, the incompressibility condition (1.2), allows us to obtain a closed form as follows. Take the divergence of (1.1) to obtain

$$-\Delta p = \nabla \cdot (v \cdot \nabla) v = \partial_{x_i} v_j \partial_{x_j} v_i.$$

Then, by taking the inverse Laplacian, we may write (1.1) as

$$\partial_t v + (v \cdot \nabla) v = \Delta v + \nabla \Delta^{-1} (\partial_{x_i} v_j \partial_{x_j} v_i), \tag{1.4}$$
$$\nabla \cdot v = 0 \tag{1.5}$$
$$v(0) = v_0. \tag{1.6}$$

More importantly for us, this reconstruction operation can be written as

$$p = K * (v \otimes v) \tag{1.7}$$

where $K(x,y)$ is the kernel of $(-\Delta)^{-1}(\text{div div})$. When the derivatives coincide, i.e. terms of the form $\Delta^{-1}\partial_{x_j}\partial_{x_j}$, we have a delta-function contribution, which is bounded from $H^2 \to L^2$. From the other terms, we obtain a singular integral operator. This allows us to pass a-priori estimates on the velocity to the pressure.

Second, we note how the equations scale. In particular, for $\lambda > 0$ and v a solution to (1.1), define

$$v_\lambda = \frac{1}{\lambda} v\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right),$$
$$v_0^\lambda = \frac{1}{\lambda} v\left(\frac{x}{\lambda}\right),$$

so that v_λ is a solution to (1.1) with initial data v_0^λ.

We are interested in self-similar solutions. For (1.1), that exactly means a solution pair v should be invariant under the scaling the equations, i.e. $v = v_\lambda$ for each $\lambda > 0$. Now, if v is a self-similar solution to (1.1) on $[0, T) \times \mathbb{R}^3$, it is clear that v can be extended to a self-similar solution on $[0, \infty) \times \mathbb{R}^3$ via

$$v(t, x) = \frac{1}{\lambda} v\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$$

where the right hand side is defined for $\lambda^2 > t/T$. So, self-similar solutions are always global.

Clearly, in order for an initial data v_0 to have a self-similar solution in any reasonable sense, we must have

$$v_0(x) = v_0^\lambda(x) = \lambda^{-1} v(\lambda^{-1} x).$$

In other words, an initial data v_0 ought to be -1-homogeneous for us to expect existence.

If we are interested in what restrictions on we must place on an initial data to guarantee existence, it is natural to try and find a norm which will be invariant under the scaling $v \mapsto v_\lambda$, i.e. a critical space for (1.1). In particular $v = v_\lambda$, so the only spaces, which contain self-similar solutions are critical spaces.

By a simple calculation, one notices that $\|v_\lambda\|_{L^3}$ is independent of λ and thus $L^3(\mathbb{R}^3)$ is the critical Lebesgue space for NSE. In L^3, we have the following classical local well-posedness theorem of Kato ([3]):

Theorem 1.1 (Kato). *Let $u_0 \in L^3$ be a divergence free (in the sense of distributions) vector field on $\mathbb{R}^3$. Then, there exists ϵ_0 a universal constant and $T = T(u_0)$, such that there exists a unique divergence-free mild solution to (1.1), $u \in C([0, T); L^3)$ and if $\|u_0\|_{L^3} \leq \epsilon_0$, we may take $T = \infty$. That is, u satisfies (1.1) in the sense that*

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \left[K * (u \otimes u) + \nabla \cdot (u \otimes u)\right](s) \, \mathrm{d}\, s$$

for each $0 \leq t < T$, where the equality is in the sense of $L^3(\mathbb{R}^3)$.

So, in L^3, a critical space, we have local existence and uniqueness to (1.1) for any large initial data. In fact, L^3 is the largest critical space for which local well-posedness is known for all initial data. However, L^3 does not contain any -1-homogeneous initial data. Indeed, $|x|^{-1} \notin L^3(\mathbb{R}^3)$. But, we do have $|x|^{-1}$ in the weak space $L^{3,\infty}(\mathbb{R}^3)$. It is easy to verify that $L^{3,\infty}(\mathbb{R}^3)$ satisfies that same scaling as $L^3(\mathbb{R}^3)$ and is also a critical space for (1.1). However, it is strictly larger than $L^3(\mathbb{R}^3)$.

The following theorem of Hao Jia and Vladimir Sverak ([2]) states that this (along with some regularity) is actually sufficient to guarantee the existence of self-similar solution:

Theorem 1.2 (Hao/Sverak). *Let v_0 be -1-homogeneous ($v_0 = v_0^\lambda$ for each $\lambda > 0$) and smooth away from 0. Then, there exists at least one v smooth away from $(t, x) = (0, 0)$, a self-similar solution to (1.1) with initial data v_0.*

We will state later exactly in what sense v is a solution to (1.1). But, it is already possible to see why this theorem is interesting. First, note that since v_0 is -1-homogeneous, it is completely determined by its restriction to S^2, where it is smooth and hence bounded. In particular, $\|v_0|_{S^2}\|_{L^\infty} = \|v_0\|_{L^{3,\infty}}$ seems like a natural way to measure such initial data. However, there is no size restriction on v_0. This is an existence result for large initial data in a critical space $L^{3,\infty}$ which is strictly larger than L^3.

Second, the restrictions on the initial data are very mild. In some sense, Hao and Sverak show that the bare minimum restriction on v_0 plus some regularity, necessarily implies existence of a (smooth) self-similar solution.

Remark. Note that unlike Kato's local well-posedness theorem, we do not get uniqueness from Hao and Sverak's theorem. Indeed, in general we do not even expect uniqueness to be true for large initial data. The difference comes from the use of the Banach fixed point theorem (contraction mapping principle) in the local well-posedness as opposed to the Leray-Schauder continuation theorem in Hao/Sverak. The Leray-Schauder theorem is a variant of the Schauder fixed point theorem, which like the finite dimensional version Brouwer's fixed point theorem, definitely says nothing about uniqueness.

In the second section, we introduce a few preliminary results concerning the 3D Navier-Stokes equations, define what we mean by a solution, and introduce our abstract technique for proving existence. We then conclude in the third section with a sketch of the proof of Hao and Sverak's theorem.

2 Preliminaries

2.1 Energy inequalities and Leray Solutions

For a classical solution $v(t,x) \in C^\infty([0,\infty) \times \mathbb{R}^3)$ to (1.1) which vanishes sufficiently rapidly, we have the nonlinearity paired with itself vanishes since:

$$\begin{aligned}\int_{\mathbb{R}^3} (v \cdot \nabla) v \cdot v &= \int_{\mathbb{R}^3} v_i v_j \partial_{x_i} v_j \\ &= -\int_{\mathbb{R}^3} v_j^2 \partial_{x_i} v_i + v_j v_i \partial_{x_i} v_j \\ &= -\int_{\mathbb{R}^3} (v\nabla) v \cdot v.\end{aligned}$$

Thus, we obtain the following energy equality:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|v(t)\|_{L^2}^2 &= \int_{\mathbb{R}^3} v \cdot \partial_t v \\ &= \int_{\mathbb{R}^3} v \cdot (\nabla p) - [(v \cdot \nabla)\, v] \cdot v + v \cdot \Delta v \\ &= \int_{\mathbb{R}^3} v_i \partial_{x_i} p + v_i \partial_{x_j x_j} v_i \\ &= -\int_{\mathbb{R}^3} (p \partial_{x_i} v_i) + \partial_{x_j} v_i \partial_{x_j} v_i \\ &= -\int_{\mathbb{R}^3} |\nabla v|^2\end{aligned}$$

Or, integrating,

$$\|v(t)\|_{L^2}^2 + 2\int_0^t \|\nabla v(s)\|_{L^2}^2 \,\mathrm{d}\,s = \|v(0)\|_{L^2}^2. \tag{2.1}$$

Thus, it is natural for us to require a weak solution to (1.1) to be in $L^2([0,T);\dot{H}^1) \cap L^\infty([0,T);L^2)$. However, since we are working with functions that look like $|x|^{-1}$, we can't quite expect that level of integrability from our solutions. But, we can expect at least that our solutions are locally (uniformly) finite in these norms. In particular, we define a solution to (1.1) as:

Definition 2.1 (Leray Solution)**.** A function $v \in L^2_{loc}([0,\infty) \times \mathbb{R}^3)$ is said to be a Leray solution to (1.1) with initial data $v_0 \in L^2_{loc}(\mathbb{R}^3)$ if

- There exists a distribution p such that equation (1.1) is satisfied by (v,p) in the sense of distributions.
- v is locally uniformly $L^\infty L^2 \cap L^2 \dot{H}^1$, in the sense that for each $R > 0$,

$$\sup_{0 \le t < R^2} \left[\sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |v(t,x)|^2 \,\mathrm{d}\,x \right] + \sup_{x_0 \in \mathbb{R}^3} \int_0^{R^2} \int_{B_R(x_0)} |\nabla v(t,x)|^2 \,\mathrm{d}\,x \,\mathrm{d}\,t < \infty \tag{2.2}$$

- Moreover, v decays at infinity sufficiently fast so that we may make sense of the pressure reconstruction (1.7), in particular,

$$\lim_{|x_0| \to \infty} \int_0^{R^2} \int_{B_R(x_0)} |v(t,x)|^2 \,\mathrm{d}\,x \,\mathrm{d}\,t = 0$$

for each $R > 0$.

- v is divergence-free in the sense of distributions.
- v obtains the initial data in the sense that

$$\|v(t) - v_0\|_{L^2(K)} \to 0$$

for each $K \subset \mathbb{R}^3$ compact.

- Finally, v is "suitable" in the sense that it satisfies a weaker, local version of the energy inequality. Namely, for each $0 < t < \infty$ and $\phi \in C_C^\infty((0,\infty) \times \mathbb{R}^3)$ with $\phi \geq 0$, we have

$$\int_{\mathbb{R}^3} |v(t)|^2 \phi + 2\int_0^t \int_{\mathbb{R}^3} |\nabla v|^2 \phi \leq \int_0^t \int_{\mathbb{R}^3} |v|^2 (\partial_t \phi + \Delta \phi) + \int_0^t \int_{\mathbb{R}^3} (|v|^2 + 2p) v \cdot \nabla \phi. \tag{2.3}$$

Basically, these are the weak solutions to (1.1) which aren't horribly whacked. Moreover, they still come with an a-priori estimate resembling (2.1).

Theorem 2.1 (A-priori estimates for Leray Solutions)**.** *Let $v_0 \in L^2_{unif}(\mathbb{R}^3)$ and v a Leray Solution starting from v_0. Then, for any fixed $R > 0$, there exists a $\lambda > 0$ such that*

$$\sup_{0 \leq t < \lambda R^2} \left[\sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |v(t)|^2 \right] + 2 \sup_{x_0 \in \mathbb{R}^3} \int_0^{\lambda R^2} \int_{B_R(x_0)} |\nabla v|^2 \leq C \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |v_0|^2. \tag{2.4}$$

As a reference for Leray solutions, pretty much the only exposition of them I could find is the book ([4]). For general information about weak solutions, I have found the book ([6]) to be a good resource.

2.2 Leray-Schauder Degree Theory

The main soft analysis result we will use here to prove existence is the following:

Theorem 2.2 (Leray-Schauder Continuation)**.** *Suppose X is a Banach space, $F : X \to X$ is compact with a unique fixed point $F(x^*) = x^*$ and (Frechet) differentiable at x^* with $D_{x^*}[Id - F] = Id \in GL(X)$, and $G : X \to X$ is compact. Further suppose $H : [0,1] \times X \to X$ with $H(0) = F$ and $H(1) = G$ such that:*

- *A-priori bounds - there exists a constant C independent of t, such that $H(t,x) = x$ implies $\|x\|_X \leq C$*
- *Compactness - H is compact with respect to the product topology on $[0,1] \times X$.*

Then, there exists a (not necessarily unique) fixed point to the problem $G(x) = x$ in X.

This is a bit abstract, so to think of why this might be useful for proving existence, think of F as a solution operator to some initial value problem with small initial data (so that the problem is well-posed) and G as the solution operator to same problem, but for large initial data. Then, this theorem says that if you can compactly deform F to G, then the well-posedness of small initial data implies existence for large initial data (but not necessarily uniqueness).

Now, this isn't exactly a standard analytic tool and is rather unintuitive the first time you see it, so I've tried to give some idea of why it's true. In particular, the standard proof uses the topological idea of degree theory.

Degree theory is basically the idea that I want that counting the number of solutions of a problem $f(x) = 0$ in some domain, is almost a topological invariant. In finite dimensions, fix $\Omega \subset \mathbb{R}^n$ compact and $f \in C^\infty(\Omega; \mathbb{R}^n)$. Further, if $D_x f$ is an isomorphism whenever $f(x) = 0$, (i.e. 0 is a regular value) f has finitely many 0's in Ω and the following sum is defined:

$$\deg(f, \Omega) = \sum_{x_i} sgn(\det Df_{x_i}).$$

This roughly counts the number of solutions to the equation $f(x) = 0$ where $x \in \Omega$ up to some orientation of the solution. For us, the important properties this satisfies are the following:

- If $\deg(f, \Omega) \neq 0$, then there exists $x \in \Omega$ such that $f(x) = 0$.
- If $0 \in \Omega$, $\deg(Id, \Omega) = 1$
- If $H : [0,1] \times \Omega \to \mathbb{R}^n$ is C^∞ with 0 a regular value, $0 \notin H([0,1], \partial\Omega)$, and $H(0) = f$ and $H(1) = g$ (such a function H is called an admissible homotopy from f to g), then

$$\deg(f, \Omega) = \deg(g, \Omega).$$

Remark. The third property shows that $\deg(f, \Omega)$ is invariant under certain continuous deformations of the function f, which makes deg into a topological invariant of the space $C^\infty(\Omega, \mathbb{R}^n)$, i.e. $\deg : C^\infty \to \mathbb{Z}$ is continuous, which we will be very useful for us. The more natural definition of simply counting solutions to $f(x) = 0$ does not satisfy this property, which is why we prefer the slightly less intuitive signed count.

Remark. For any point $p \in \mathbb{R}^n$ a regular value for f, $f(x) = p$ has finitely many solutions. Therefore, using the "continuity" of deg, by Sard's theorem and the Stone-Weierstrauss theorem, we may extend deg to be defined on all functions $f \in C(\Omega; \mathbb{R}^n)$ satisfying the same properties as above.

Now, this would be great if we were interested in studying the fixed points of finite dimensional problems. But we are more interested in the situation

$$F(x) = x$$

where $F : X \to X$ for X a Banach space. However, in this case we cannot extend deg to all continuous functions $f \in C(\Omega; X)$, but rather only functions F with finite dimensional approximations, namely compact mappings. Indeed, the Leray-Schauder degree is the extension of deg to

$$K(\Omega; X) = \{Id - C \mid C : \Omega \to X \text{ is compact}\}.$$

The extension also written as deg, satisfies slight variants of the above properties:

- If $\deg(Id - F, \Omega) \neq 0$, then there exists $x \in \Omega$ such that $F(x) = x$.
- If $0 \in \Omega$, $\deg(Id, \Omega) = 1$
- If $H : [0,1] \times \Omega \to X$ is compact (with respect to the product topology on $[0,1] \times \Omega$), continuous with $0 \notin H(0, \partial\Omega)$, and $H(0) = Id - F$ and $H(1) = Id - G$, then

$$\deg(Id - F, \Omega) = \deg(Id - G, \Omega).$$

Given this degree function on $K(\Omega; X)$ with the above properties, it is quite easy to prove the Leray-Schauder theorem. All the work really lies in constructing such a degree.

Leray-Schauder Continuation. Since $H(t, x)$ satisfies the a-priori bound, we may pick $\Omega = B_C(0) \subset X$ the ball of radius C. Then, by assumption H is an admissible homotopy from the restriction of F to Ω to the restriction of G to Ω. Hence,

$$\deg(Id - F, \Omega) = \deg(Id - G, \Omega).$$

On the other hand,

$$\deg(Id - F, \Omega) = 1,$$

since returning to our original definition counts the one solution to the problem $x - F(x) = 0$ in Ω and we have chosen the sign at x^* by picking the orientation $D_{x^*}[Id - F] = Id$. In particular,

$$\deg(Id - G, \Omega) \neq 0$$

gives $G(x) = x$ for some $x \in \Omega$. $\square$

3 Proof Sketch

3.1 Step One: Functional set up

Now, we want to pick the proper spaces and operators to apply the Leray-Schauder Continuation Theorem. First, we use self-similarity to reduce the problem and find the object which should be a fixed point. For any $u(t,x)$, a Leray solution with initial data u_0, is completely determined by its profile at time $t=1$, $U(x)=u(1,x)$ via

$$u(x,t)=\frac{1}{\sqrt{t}}U\left(\frac{x}{\sqrt{t}}\right),$$

where $U(x)$ satisfies an elliptic equation $\mathcal{G}\,U=0$ and has decay at infinity $|U(x)|\lesssim |x|^{-1}$. In fact, $u(t,x)$ is a self-similar solution to NSE if and only if $\mathcal{G}\,U=0$ and $|U(x)|\lesssim|x|^{-1}$.

Further, we may decompose

$$U(x)=e^{\Delta}u_0+V.$$

where V then satisfies the elliptic equation

$$-\Delta V-\frac{V}{2}-\frac{x}{2}\cdot\nabla V+\nabla P=\mathcal{L}(V,u_0) \tag{3.1}$$

$$\nabla\cdot V=0 \tag{3.2}$$

where

$$\mathcal{L}(u_0,V)=-\left[V\cdot\nabla V+e^{\Delta}u_0\cdot\nabla V+\nabla e^{\Delta}u_0\cdot V+e^{\Delta}u_0\cdot\nabla e^{\Delta}u_0\right]$$

and $P(x)=p(x,1)$. Let $S:X\to X$ be the solution operator to the above, so that $S(f)$ satisfies (3.1) with right hand side f. Then, define $H(\mu,V)=S(\mathcal{L}(V,\mu u_0))$. Further, pick X to be the Banach space

$$X=\{f\in C^1(\mathbb{R}^3;\mathbb{R}^3)\mid \nabla\cdot f=0,\|f\|_X<\infty\}$$

where the norm is given by

$$\sup_{x\in\mathbb{R}^3}\left[(1+|x|^2)|f(x)|+(1+|x|^3)|\nabla f(x)|\right].$$

We claim this is the functional analytic set-up want to work in. There are many things to verify:

Lemma 3.1. *There is a well-defined function $H(\mu,V):[0,1]\times X\to X$ which is obtain via the composition $S\circ\mathcal{L}$.*

Lemma 3.2. *For $u(t,x)$ and $V(x)$ related as above, u is a self-similar Leray solution with initial data μu_0, if and only if $H(\mu,V)=V$.*

Together, these two claims give that if we can apply the Leray-Schauder Continuation theorem to H, then there exists a self-similar solution to (1.1) and the proof is complete. It will still remain to show that the hypotheses of the Leray-Schauder theorem apply to H. That is exactly the content of the following three claims:

Lemma 3.3. *There exists a unique $V^*\in X$ such that $H(0,V^*)=V^*$ and $D_{V^*}[H(0,\cdot)]=0$.*

Lemma 3.4. *There exists a constant $C=C(u_0)$ independent of μ such that if $H(\mu,V)=V$, then $\|V\|_X\le C$.*

Lemma 3.5. *The map $H(\mu,V):[0,1]\times X\to X$ is continuous and, in particular, compact.*

We shall now prove claim 1: It suffices to show that there exists such an operator $S:X\to X$. We isolated the terms on the left hand side of (3.1) because in normal coordinates this becomes a forced Stokes system,

$$\partial_t v-\Delta v+\nabla p=t^{-3/2}F\left(\frac{x}{\sqrt{t}}\right) \tag{3.3}$$

$$\nabla\cdot v=0 \tag{3.4}$$

$$v(0,x)=0 \tag{3.5}$$

with $V(x)=v(1,x)$. Stokes systems have been studied and the following is true:

Theorem 3.6. *Suppose $v \in L^\infty_t L^\gamma_x$ for some $\gamma > 1$. If v solves (3.3) for some distribution p, then if $F \leq \frac{C_1}{1+|x|^3}$, v is uniquely defined as the mild solution,*

$$v(t,x) = \int_0^t e^{(t-s)\Delta}\, \mathbb{P}\, s^{-3/2} F(s^{-1/2}x)\, \mathrm{d}\, s$$

where $\mathbb{P}$ denotes the projection onto the subspace of divergence-free vector fields. Moreover, if $V(x) = v(1,x)$,

$$\|V\|_X \lesssim C_1 \qquad \text{and} \qquad \|V\|_{C^{1,\alpha}(B_R(0))} \lesssim_{\alpha,R} C_1.$$

Clearly, $V \in X$ implies $v(t,x) = t^{-1/2}V(t^{-1/2}x) \in L^\infty L^p$ so that the Stokes system has a well-defined solution. Now, since $\mathcal{L}(V,\mu) \in X$ for $V \in X$, it follows that $H(V,\mu)$ is well defined on X. Since $\mathcal{L}(V,\mu)$ actually has decay like $|x|^{-3}$, when $V \in X$, we obtain $H(V,\mu)$ maps X into X.

And now, claim 2: If $V \in X$ is a fixed point $H(1,V) = V$, then clearly V satisfies the above elliptic PDE (3.1), which implies $\mathcal{G}\,U = 0$ and has the requisite decay. If u is a self-similar Leray solution with initial data u_0, it follows that V satisfies the elliptic PDE (3.1) above. It remains only to show that $V \in X$. We will show this and claim 3 simultaneously by proving an a-priori bound for H.

3.2 Step Two: A priori bounds

Suppose $u(t,x)$ is a self-similar Leray solution with initial data u_0. Then, it is clear that $|U(x)| \leq C(1+|x|)^{-1}$. However, when we look at U as a perturbation of the heat evolution, we gain two powers of $|x|$ in decay. Indeed, we have the following improvement:

$$|\partial_\alpha(U(x) - e^{\Delta}u_0)| \leq \frac{C(\alpha,u_0)}{1+|x|^{3+\alpha}},$$

which is actually the whole point of the functional set-up. Moreover, U is smooth.

We proceed as follows. For any "good" point of the initial data, i.e. not 0, u_0 is smooth nearby. Suppose this smoothness around propagates forward for some small time uniformly for any Leray solution starting from u_0. In particular, we mean for any good point there exists a $T(u_0) > 0$ such that for any u a Leray solution starting from u_0 we have

$$\|\partial_x^\alpha \partial_t^\beta\|_{L^\infty(B\times[0,T])} \leq C(\|u_0\|_{L^\infty(B)})$$

Then, we can cover $S^2 \subset \mathbb{R}^3$ by balls $B_{1/2}(x_0)$, for which we have a uniform estimate of $\|u_0\|_{L^\infty}$ and in particular,

$$\|\partial_x^\alpha \partial_t\|_{L^\infty(B_{1/2}(x_0)\times[0,T])}$$

is uniformly bounded independent of x_0. Therefore, by the mean value theorem, we can write

$$|u(t,x) - u_0| \leq Ct$$

for $1/2 \leq |x| \leq 3/2$ and $0 \leq t \leq T$. But, for any $\lambda > 0$, u_λ is a Leray solution starting from u_0, and satisfies the same estimate by uniformity over all Leray solutions.

$$|\lambda^{-1}u(t\lambda^{-2}, x\lambda^{-1}) - u_0(x)| \leq Ct$$

Now, picking $\lambda = \sqrt{t}$ and using -1-homogeneity of u_0,

$$|u(1, xt^{-1/2}) - u_0(xt^{-1/2})| \leq Ct^{3/2}.$$

But, this just

$$|U(y) - u_0(y)| \leq C|y|^{-3}$$

for $|y| \geq 1/(2T)$. Since the heat profile at time 1 satisfies the same scaling, we also have

$$|e^{\Delta}u_0(y) - u_0(y)| \leq C|y|^{-3}$$

for $|y| \geq 1/(2T)$. On the other hand, we can use our a-priori estimates for Leray solutions to give $U \in H^1(B_{1/T}(x))$. Since U satisfies an elliptic equation, elliptic estimates give U is smooth and bounded for $|y| \leq 3/(2T)$. Therefore, boundedness of the heat profile gives

$$|U(y) - e^{\Delta}u_0(y)| \leq \frac{C}{1+|y|^3}$$

for each $y \in \mathbb{R}^3$. We can do the same estimate on the derivatives of U to obtain the desired improvement.

The key property we used was that Leray solutions are nice enough to propagate regularity for short time which is interesting in and of itself.

3.3 Step Three: Compactness and Well-Posedness for small μ

In this case, compactness comes from the properties of the solution operator S to the Stokes system (3.3). In particular, if $f \in C^1$, $S(f)$ is in $C^{1,\alpha}_{loc}$ and we gain regularity. Moreover, if f decays like $|x|^{-4}$, which $\mathcal{L}(V,\mu)$ does, then $S(f)$ decays like $|x|^{-3}$, so that we gain decay. Therefore, $H(\mu, V) : X \times [0,1] \to X$ is a compact map.

Finally, unique solvability of $H(\mu, V) = V$ for $\mu = 0$ follows from uniqueness to the Navier-Stokes equations in X for small data. Also, we compute the derivative at $V = 0$, the unique fixed point for $\mu = 0$,

$$D_0H(0,V) = D_0S(V \cdot \nabla V) = D_0S(V \cdot \nabla 0 + 0 \cdot \nabla V) = D_0S(0) = 0.$$

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