Tenth Homework: MATH 410
Due Monday, 5 November 2018

1. Exercise 1 of Section 3.4 of the text.
2. Exercise 5 of Section 3.4 of the text.
3. Show that $f : \mathbb{R}_+ \to \mathbb{R}$ given by $f(x) = \sin(1/x)$ is not uniformly continuous.
4. Prove Proposition 8.6 on page 46 in the notes.
5. Give a counterexample to each of the following false assertions.
   (a) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and increasing over $\mathbb{R}$ then $f' > 0$ over $\mathbb{R}$.
   (b) If $f : (a, b) \to \mathbb{R}$ is continuous then f has a minimum or a maximum over (a, b).
6. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be uniformly continuous over D. Let $\{x_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence contained in D. Show that $\{f(x_k)\}_{k\in\mathbb{N}}$ is a convergent sequence.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove it is continuous.
8. Let $f(x) = \sinh(x)$ for every $x \in \mathbb{R}$. Show that
$$\sinh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\, x^{2k+1} \quad \text{for every } x \in \mathbb{R}\,.$$
9. Evaluate the following limit. Give your reasoning.
$$\lim_{x\to 3} \frac{x^4 - 81}{x^2 - 9}\,.$$
10. Suppose that $f : (a, b) \to \mathbb{R}$ is twice differentiable and that $f'' : (a, b) \to \mathbb{R}$ is bounded over (a, b). Show that there exists an $M \in \mathbb{R}_+$ such that for every $x, y \in (a, b)$ we have
$$\big|f'(x) - f'(y)\big| \le M\,|x - y|\,.$$
11. Prove that for every $x > 0$ we have
$$1 + \tfrac{3}{2}x < (1 + x)^{\frac{3}{2}} < 1 + \tfrac{3}{2}x + \tfrac{3}{8}x^2\,.$$
12. Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0, 1]$ if there exists a $C \in \mathbb{R}_+$ such that for every $x, y \in D$ we have
$$|f(x) - f(y)| \le C\,|x - y|^{\alpha}\,.$$
Show that if $f : D \to \mathbb{R}$ is Hölder continuous of order α for some $\alpha \in (0, 1]$ then it is uniformly continuous over D.
13. Let $\alpha \in (0, 1)$. Let $f : [0,\infty) \to \mathbb{R}$ be defined by $f(x) = x^{\alpha}$. Show that f is uniformly continuous over $[0,\infty)$. Hint: Use the previous problem after showing that
$$\big|x^{\alpha} - y^{\alpha}\big| \le |x - y|^{\alpha} \quad \text{for every } x, y \in [0,\infty)\,.$$
14. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the equation $f'(x) = 0$ has at most one solution over $x \in \mathbb{R}$. Show the equation $f(x) = 0$ has at most two solutions over $x \in \mathbb{R}$.
15. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Write negations of the following assertions.
   (a) "For all sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ contained in D we have
$$\lim_{k\to\infty} |x_k - y_k| = 0 \quad\Longrightarrow\quad \lim_{k\to\infty} \big|f(x_k) - f(y_k)\big| = 0\,."$$
   (b) "For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all points $x, y \in D$ we have
$$|x - y| < \delta \quad\Longrightarrow\quad \big|f(x) - f(y)\big| < \epsilon\,."$$