Fifth Homework: MATH 410
Due Tuesday, 3 October 2017

1. Exercise 2 of Section 2.2 in the text.
2. Exercise 4 of Section 2.2 in the text.
3. Prove Proposition 4.1 in the notes.
4. Prove Proposition 4.2 in the notes.
5. Prove Proposition 4.6 in the notes.
6. Prove Proposition 4.8 in the notes.
7. Prove Proposition 4.9 in the class notes.
8. Prove Proposition 4.12 in the class notes.
9. Prove Proposition 4.13 in the class notes.
10. Do the exercise on page 59 of the class notes.
11. Consider the real sequence $\{b_k\}_{k\in\mathbb{N}}$ given by

$$b_k = (-1)^k \left(3 + \frac{1}{(k+1)^2} \right) \quad \text{for every } k \in \mathbb{N},$$

where $\mathbb{N} = \{0, 1, 2, \cdots\}$.
   (a) Give the first three terms of the subsequence $\{b_{3k}\}_{k\in\mathbb{N}}$.
   (b) Give the first three terms of the subsequence $\{b_{2^k-1}\}_{k\in\mathbb{N}}$.
   (c) Compute $\limsup_{k\to\infty} b_k$ and $\liminf_{k\to\infty} b_k$. Justify your answers.
12. Determine all the values of $a \in \mathbb{R}$ for which

$$\sum_{n=2}^{\infty} \frac{1}{\log(n)}\, a^n \quad \text{converges}.$$

13. Determine all the values of $a \in \mathbb{R}$ for which

$$\sum_{k=0}^{\infty} \left(\frac{2k+3}{k^4+1} \right)^a \quad \text{converges}.$$

14. Determine all the values of $a \in \mathbb{R}$ for which

$$\sum_{m=1}^{\infty} \frac{1}{m^2} \left(2 + (-1)^m \right)^m a^m \quad \text{converges}.$$

15. Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$.
    Write the negations of the following assertions.
    (a) "For every $m \in \mathbb{R}$ one has $b_j > m$ frequently as $j \to \infty$."
    (b) "Every sequence in A has a subsequence that converges to a limit in A."