Third Homework: MATH 410
Due Wednesday, 19 September 2018

1. Let $\{a_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$. Show that every subsequence of $\{a_n\}_{n\in\mathbb{N}}$ is convergent and that they all have the same limit.
2. Show that $\{1/n\}_{n\in\mathbb{Z}_+}$ is not contracting.
3. Let $a_0 > 0$ and $a_{n+1} = 1/(1+a_n^2)$ for every $n \in \mathbb{N}$. Show that $\{a_n\}_{n\in\mathbb{N}}$ is contracting.
4. Let $a_0 > 0$ and $a_{n+1} = 1/(1+a_n)$ for every $n \in \mathbb{N}$. Show that $\{a_n\}_{n\in\mathbb{N}}$ is contracting and evaluate
$$\lim_{n\to\infty} a_n \,.$$
5. Exercise 12 of Section 2.4 in the text.
6. Exercise 1 of Section 9.1 in the text.
7. Exercise 2 of Section 9.1 in the text.
8. Exercise 3 of Section 9.1 in the text.
9. Exercise 4 of Section 9.1 in the text.
10. Consider the infinite series
$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)} \,.$$
Find an expression for its partial sums. Use this expression to determine if the series converges or diverges. If it converges then find its sum.
11. Consider a formal infinite series of the form
$$\sum_{k=1}^{\infty} kr^k \,,$$
for some $r \in \mathbb{R}$. Find all the values of r for which this series converges and evaluate the sum. (Hint: Find an explicit expression for the partial sums and evaluate the limit. The explicit expression may be derived from the analogous expression for a geometric series.)
12. The proof of Proposition 3.4 in the notes argues that the direct comparison test applies whenever the limit comparison test applies, and that the limit comparison test applies whenever the ratio comparison test applies. Can you find (a) an example where the direct comparison test applies but the limit comparison test fails, and (b) an example where the limit comparison test applies but the ratio comparison test fails?