Second In-Class Exam Solutions
Math 410, Professor David Levermore
Friday, 2 November 2018

1. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Write negations of the following assertions.
   (a) "For all sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ contained in D we have
$$\lim_{k\to\infty} |x_k - y_k| = 0 \qquad \Longrightarrow \qquad \lim_{k\to\infty} \big|f(x_k) - f(y_k)\big| = 0\,.\text{"}$$
   (b) "For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all points $x, y \in D$ we have
$$|x - y| < \delta \qquad \Longrightarrow \qquad \big|f(x) - f(y)\big| < \epsilon\,.\text{"}$$

Solution (a). There exists sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ contained in D such that
$$\lim_{k\to\infty} |x_k - y_k| = 0 \qquad \text{and} \qquad \limsup_{k\to\infty} \big|f(x_k) - f(y_k)\big| > 0\,.$$

Solution (b). There exists $\epsilon_o > 0$ such that for every $\delta > 0$ there exists $x_\delta, y_\delta \in D$ such that
$$|x_\delta - y_\delta| < \delta \qquad \text{and} \qquad \big|f(x_\delta) - f(y_\delta)\big| \geq \epsilon_o\,.$$

2. [10] Give (with reasoning) a counterexample to each of the following false assertions.
   (a) If $f : (a, b) \to \mathbb{R}$ is continuous then f has a minimum or a maximum over (a, b).
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative $f' : \mathbb{R} \to \mathbb{R}$ is continuous.

Solution (a). There are many easy counterexamples. The simplest might be
$$f : (-1, 1) \to \mathbb{R} \quad \text{given by} \quad f(x) = x\,.$$
This is clearly continuous over $(-1, 1)$ and has no minimum or maximum over $(-1, 1)$ because for every $x \in (-1, 1)$ we have
$$\inf\big\{f(y) \,:\, y \in (-1, 1)\big\} = -1 < f(x) < 1 = \sup\big\{f(y) \,:\, y \in (-1, 1)\big\}\,.$$

Solution (b). The example from the notes and lecture is defined for every $x \in \mathbb{R}$ by
$$f(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right) & \text{for } x \neq 0\,, \\ 0 & \text{for } x = 0\,. \end{cases}$$
This function is differentiable over $\mathbb{R}$ with
$$f'(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) + 2x \cos\left(\dfrac{1}{x}\right) & \text{for } x \neq 0\,, \\ 0 & \text{for } x = 0\,. \end{cases}$$
But this derivative is not continuous at $x = 0$ because
$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \sin\left(\frac{1}{x}\right) \qquad \text{does not exist}\,.$$

3. [15] Let $f : (a,b) \to \mathbb{R}$ be differentiable at a point $c \in (a,b)$ with $f'(c) < 0$. Show that there exists a $\delta > 0$ such that

$$x \in (c-\delta, c) \subset (a,b) \implies f(x) > f(c)\,,$$
$$x \in (c, c+\delta) \subset (a,b) \implies f(c) > f(x)\,,$$

Remark. You are being asked to prove the second part of the Transversality Lemma (Proposition 6.2).

Solution. Because f is differentiable at c we know that

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}\,.$$

By the ϵ-δ definition of limit, this means that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in (a,b)$ we have

$$0 < |x-c| < \delta \qquad \implies \qquad \left| \frac{f(x)-f(c)}{x-c} - f'(c) \right| < \epsilon\,.$$

Because $f'(c) < 0$ we may take $\epsilon = -f'(c)$ above to conclude that there exists $\delta > 0$ such that for every $x \in (a,b)$ we have

$$0 < |x-c| < \delta \qquad \implies \qquad \left| \frac{f(x)-f(c)}{x-c} - f'(c) \right| < -f'(c)\,.$$

Because $c \in (a,b)$ we may assume that δ is small enough so that $(c-\delta, c+\delta) \subset (a,b)$. Then we have

$$0 < |x-c| < \delta \qquad \implies \qquad 2f'(c) < \frac{f(x)-f(c)}{x-c} < 0\,.$$

This implies that $x-c$ and $f(x)-f(c)$ will have opposite signs when $0 < |x-c| < \delta$. It follows that

$$x \in (c-\delta, c) \quad \implies \quad x-c < 0 \quad \implies \quad f(x)-f(c) > 0\,,$$
$$x \in (c, c+\delta) \quad \implies \quad x-c > 0 \quad \implies \quad f(x)-f(c) < 0\,.$$

Because $(c-\delta, c) \subset (a,b)$ and $(c, c+\delta) \subset (a,b)$, the result follows. $\square$

4. [15] If $f(x) = \cos(x)$ for every $x \in \mathbb{R}$ then for every $k \in \mathbb{N}$ we have

$$f^{(2k)}(x) = (-1)^k \cos(x)\,, \qquad f^{(2k+1)}(x) = (-1)^{k+1} \sin(x) \qquad \text{for every } x \in \mathbb{R}\,.$$

Use this fact to show that

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} \quad \text{for every } x \in \mathbb{R}\,.$$

Remark. There are many convergence tests that can be applied to show that the above series converges absolutely. For example, if we apply the Ratio Test then because for every $x \in \mathbb{R}$ we have

$$\lim_{k\to\infty} \frac{\dfrac{1}{(2k+2)!}\,|x|^{2k+2}}{\dfrac{1}{(2k)!}\,|x|^{2k}} = \lim_{k\to\infty} \frac{|x|^2}{(2k+1)(2k+2)} = 0\,,$$

we conclude for every $x \in \mathbb{R}$ that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} \qquad \text{converges absolutely}\,.$$

However, such convergence tests do not show that the series converges to $\cos(x)$, which is what you are being asked to show!

Solution. Because for every $k \in \mathbb{N}$ we have

$$f^{(2k)}(0) = (-1)^k\,, \qquad f^{(2k+1)}(0) = 0\,,$$

the series is just the formal Taylor series for f centered at 0. The n^{th} partial sum of this series can be expressed as a Taylor polynomial approximation in two ways:

$$\sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}\, x^{2k} = T_0^{2n} \cos(x) = T_0^{2n+1} \cos(x)\,.$$

If we use the last expression then the Lagrange Remainder Theorem states that for every $x \in \mathbb{R}$ there exists some p between 0 and x such that

$$\cos(x) = T_0^{2n+1} \cos(x) + \frac{(-1)^{n+1}}{(2n+2)!} \cos(p) x^{2n+2}\,.$$

Hence, because $|\cos(p)| \leq 1$ for every $p \in \mathbb{R}$, we have the bound

$$\left| \cos(x) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}\, x^{2k} \right| \leq \frac{1}{(2n+2)!}\, |x|^{2n+2}\,.$$

However, either because factorials grow faster than exponentials, or because the series converges, for every $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{(2n+2)!}\, |x|^{2n+2} = 0\,.$$

Therefore the sequence of partial sums converges to $\cos(x)$. □

5. [15] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose that f' is increasing over a bounded interval (a, b). Prove that f is strictly convex over $[a, b]$.

Remark. You are being asked to prove part of the Convexity Characterization Theorem (Proposition 7.6) — specifically, direction ($\Longrightarrow$) of characterization (i).

Solution. Let $x, y, z \in [a, b]$ with $x < y < z$. Because f is differentiable over $\mathbb{R}$, it is continuous over $[x, y]$ and $[y, z]$, and is differentiable over (x, y) and (y, z). Then by the Lagrange Mean-Value Theorem there exists $p \in (x, y)$ and $q \in (y, z)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(p) \,, \qquad \frac{f(x) - f(y)}{z - y} = f'(q) \,.$$

Because f' is increasing over $[a, b]$ and $a \leq x < p < y < q < z \leq b$, we know that $f'(p) < f'(q)$. Therefore

$$\frac{f(y) - f(x)}{y - x} < \frac{f(x) - f(y)}{z - y} \,.$$

Because this holds for every $x, y, z \in [a, b]$ with $x < y < z$, the difference quotient characterization of strict convexity (Proposition 6.8) implies that f is strictly convex over $[a, b]$. $\square$

6. [15] Prove that for every nonzero $x \in \mathbb{R}$ we have

$$1 + \tfrac{6}{5}x < (1 + x)^{\frac{6}{5}} \,.$$

Solution. One approach to this problem uses the Monotonicity Theorem. Define $g(x) = (1 + x)^{\frac{6}{5}} - 1 - \frac{6}{5}x$ for every $x \in \mathbb{R}$. Then g is continuously differentiable with

$$g'(x) = \tfrac{6}{5}\big[(1 + x)^{\frac{1}{5}} - 1\big] \,.$$

Clearly, $g'(x) < 0$ for $x < 0$ while $g'(x) > 0$ for $x > 0$. By the Monotonicity Theorem, g is decreasing over $(-\infty, 0]$ and g is increasing over $[0, \infty)$. Therefore the global minimum of g over $\mathbb{R}$ is $g(0) = 0$. Hence, for every $x \in \mathbb{R}$ we have

$$(1 + x)^{\frac{6}{5}} - 1 - \tfrac{6}{5}x = g(x) \geq g(0) = 0 \,.$$

The result follows. $\square$

Second Solution. Another approach to this problem uses convexity ideas. Define $f(x) = (1 + x)^{\frac{6}{5}}$ for every $x \in \mathbb{R}$. Then f is continuously differentiable over $\mathbb{R}$ with

$$f'(x) = \tfrac{6}{5}(1 + x)^{\frac{1}{5}} \,,$$

and f is twice differentiable over $\mathbb{R} - \{-1\}$ with

$$f''(x) = \tfrac{6}{25}(1 + x)^{-\frac{4}{5}} > 0 \,.$$

The Monotonicity Theorem applied to f' shows that f' is increasing over $(-\infty, -1]$ and over $[-1, \infty)$, whereby it is increasing over $\mathbb{R}$. The Convexity Characterization Theorem then implies that f is *strictly convex* over $\mathbb{R}$. This strict convexity implies that

$$f(x) - f(0) - f'(0)x > 0 \qquad \text{for every nonzero } x \in \mathbb{R} \,.$$

Therefore $(1 + x)^{\frac{6}{5}} - 1 - \frac{6}{5}x \geq 0$ for every nonzero $x \in \mathbb{R}$. The result follows. $\square$

Third Solution. Yet another approach uses the Lagrange Remainder Theorem. Define $f(x) = (1+x)^{\frac{6}{5}}$ for every $x \in \mathbb{R}$. Then f is twice differentiable over $x > -1$ with

$$f'(x) = \tfrac{6}{5}(1+x)^{\frac{1}{5}}\,, \qquad f''(x) = \tfrac{6}{25}(1+x)^{-\frac{4}{5}}\,.$$

By the Lagrange Remainder Theorem for every $x \geq -1$ there exists a p between 0 and x such that

$$f(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(p)x^2\,.$$

Hence, for every $x \geq -1$ we have

$$(1+x)^{\frac{6}{5}} - 1 - \tfrac{6}{5}x = \tfrac{3}{25}(1+p)^{-\frac{4}{5}}x^2 \geq 0\,.$$

This gives the result for every $x > -1$. We can obtain the result for every $x < -1$ by observing that in that case $(1+x)^{\frac{6}{5}} > 0$ and $(1+x) < 0$, whereby

$$(1+x)^{\frac{6}{5}} - 1 - \tfrac{6}{5}x = (1+x)^{\frac{6}{5}} + \tfrac{1}{5} - \tfrac{6}{5}(1+x) > \tfrac{1}{5} > 0\,.$$

Therefore the result follows for every $x \in \mathbb{R}$. $\square$

Remark. The Lagrange Remainder Theorem cannot be applied for $x < -1$ because $f(x)$ is not twice differentiable at $x = -1$.

Fourth Solution. A long and difficult brute force approach uses the fact that $y \mapsto y^5$ is an increasing function. Therefore

$$1 + \tfrac{6}{5}x < (1+x)^{\frac{6}{5}} \qquad \Longleftrightarrow \qquad (1+\tfrac{6}{5}x)^5 < (1+x)^6\,.$$

Two applications of the binomial expansion give

$$\begin{aligned}(1+x)^6 &= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6\,,\\ (1+\tfrac{6}{5}x)^5 &= 1 + 5(\tfrac{6}{5}x) + 10(\tfrac{6}{5}x)^2 + 10(\tfrac{6}{5}x)^3 + 5(\tfrac{6}{5}x)^4 + (\tfrac{6}{5}x)^5\\ &= 1 + 6x + \tfrac{72}{5}x^2 + \tfrac{432}{25}x^3 + \tfrac{1296}{125}x^4 + \tfrac{7776}{3125}x^5\,.\end{aligned}$$

Therefore

$$\begin{aligned}(1+x)^6 - (1+\tfrac{6}{5}x)^5 &= \tfrac{5\cdot 15 - 2\cdot 36}{5}x^2 + \tfrac{25\cdot 20 - 2\cdot 216}{25}x^3\\ &\quad + \tfrac{125\cdot 15 - 1296}{125}x^4 + \tfrac{3125\cdot 6 - 7776}{3125}x^5 + x^6\\ &= \tfrac{3}{5}x^2 + \tfrac{68}{25}x^3 + \tfrac{579}{125}x^4 + \tfrac{10974}{3125}x^5 + x^6\,.\end{aligned}$$

The result then follows upon showing that the righthand side is positive when $x \neq 0$. Because

$$(1+x)^6 - (1+\tfrac{6}{5}x)^5 = \tfrac{3}{5}x^2\big(1 + \tfrac{68}{15}x + \tfrac{579}{75}x^2 + \tfrac{10974}{1875}x^3 + \tfrac{5}{3}x^4\big)\,,$$

this is clearly equivalent to showing that

$$1 + \tfrac{68}{15}x + \tfrac{579}{75}x^2 + \tfrac{10974}{1875}x^3 + \tfrac{5}{3}x^4 > 0 \qquad \text{for every } x\,.$$

There are many ways that this might be done, but all the naive ones are difficult. For example, the local extremizers of this quartic polynomial are the roots of a cubic.

7. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the equation $f'(x) = 0$ has at most three real solutions. Prove that the equation $f(x) = 0$ has at most four real solutions.

Solution. Suppose that the equation $f(x) = 0$ has (at least) five real solutions $\{x_0, x_1, x_2, x_3, x_4\}$. Without loss of generality we can assume that

$$-\infty < x_0 < x_1 < x_2 < x_3 < x_4 < \infty\,.$$

Then for each $i = 1, 2, 3, 4$ we know that

- $f : [x_{i-1}, x_i] \to \mathbb{R}$ is differentiable (and hence is continuous),
- $f(x_{i-1}) = f(x_i) = 0$.

Rolle's Theorem then implies that for each $i = 1, 2, 3, 4$ there exists a point $p_i \in (x_{i-1}, x_i)$ such that $f'(p_i) = 0$. Because the intervals $\{(x_{i-1}, x_i)\}_{i=1}^4$ are disjoint, the points $\{p_i\}_{i=1}^4$ are distinct. Therefore equation $f'(x) = 0$ has at least four real solutions, which shows that it has more than three real solutions. □

Second Solution. Suppose that $f'(x) = 0$ has exactly m real solutions for some $m \in \{0, 1, 2, 3\}$.

If $m = 0$ then $f'(x) = 0$ has no real solutions over $\mathbb{R}$. By the Sign Dichotomy Theorem f' must be either negative or positive over $\mathbb{R}$. The Monotonicity Theorem then implies that f must be strictly monotonic (and hence one-to-one) over $\mathbb{R}$. The equation $f(x) = 0$ can thereby have at most one real solution.

If $m \in \{1, 2, 3\}$ then $f'(x) = 0$ has exactly m real solutions $\{c_1, \cdots, c_m\}$. Without loss of generality we can assume that

$$-\infty < c_1 < c_2 < \cdots < c_m < \infty\,.$$

Then by the Sign Dichotomy Theorem f' must be either negative or positive over each of the $m + 1$ disjoint intervals

$$(-\infty, c_1)\,, \quad (c_1, c_2)\,, \cdots, \quad (c_m, \infty)\,.$$

The Monotonicity Theorem then implies that f must be strictly monotonic (and hence one-to-one) over each of the $m + 1$ intervals

$$(-\infty, c_1]\,, \quad [c_1, c_2]\,, \cdots, \quad [c_m, \infty)\,.$$

Therefore the equation $f(x) = 0$ can thereby have at most one solution in each of these intervals. Because the union of these intervals is $\mathbb{R}$, the equation $f(x) = 0$ can have at most $m + 1$ real solutions. Because $m \le 3$, the equation $f(x) = 0$ can have at most 4 real solutions. □

Remark. The second solution rests upon the Sign Dichotomy Theorem and the Monotonicity Theorem. This is heavier machinery than was used in the first solution, which rests only upon Rolle's Theorem. Indeed, the Monotonicity Theorem rests upon the Mean-Value Theorem, the proof of which rests upon Rolle's Theorem.

Exercise. Modify each of the above proofs to prove the following fact. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $n \in \mathbb{Z}_+$. Suppose that the equation $f'(x) = 0$ has at most n real solutions. Show that the equation $f(x) = 0$ has at most $n + 1$ real solutions.

8. [10] Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0, 1]$ if there exists a $C \in \mathbb{R}_+$ such that for every $x, y \in D$ we have

$$|f(x) - f(y)| \leq C\,|x - y|^\alpha\,.$$

Show that if $f : D \to \mathbb{R}$ is Hölder continuous of order α for some $\alpha \in (0, 1]$ then it is uniformly continuous over D.

Solution. Let $\epsilon > 0$. Pick $\delta = (\epsilon/C)^{\frac{1}{\alpha}}$. Then, because $r \mapsto r^\alpha$ is an increasing function over $[0, \infty)$, for every $x, y \in D$ we have

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| \leq C\,|x - y|^\alpha < C\,\delta^\alpha = \epsilon\,.$$

Therefore $f : D \to \mathbb{R}$ is uniformly continuous over D. $\square$