Second In-Class Exam
Math 410, Professor David Levermore
Friday, 2 November 2018

1. [10] Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Write negations of the following assertions.
   (a) "For all sequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ contained in D we have
   $$\lim_{k\to\infty} |x_k - y_k| = 0 \quad \Longrightarrow \quad \lim_{k\to\infty} \big|f(x_k) - f(y_k)\big| = 0\,."$$
   (b) "For every $\epsilon > 0$ there exists a $\delta > 0$ such that for all points $x, y \in D$ we have
   $$|x - y| < \delta \quad \Longrightarrow \quad \big|f(x) - f(y)\big| < \epsilon\,."$$

2. [10] Give (with reasoning) a counterexample to each of the following false assertions.
   (a) If $f : (a, b) \to \mathbb{R}$ is continuous then f has a minimum or a maximum over (a, b).
   (b) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative $f' : \mathbb{R} \to \mathbb{R}$ is continuous.

3. [15] Let $f : (a, b) \to \mathbb{R}$ be differentiable at a point $c \in (a, b)$ with $f'(c) < 0$. Show that there exists a $\delta > 0$ such that
   $$x \in (c - \delta, c) \subset (a, b) \implies f(x) > f(c)\,,$$
   $$x \in (c, c + \delta) \subset (a, b) \implies f(c) > f(x)\,,$$

4. [15] If $f(x) = \cos(x)$ for every $x \in \mathbb{R}$ then for every $k \in \mathbb{N}$ we have
   $$f^{(2k)}(x) = (-1)^k \cos(x)\,, \qquad f^{(2k+1)}(x) = (-1)^{k+1} \sin(x) \qquad \text{for every } x \in \mathbb{R}\,.$$
   Use this fact to show that
   $$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} \quad \text{for every } x \in \mathbb{R}\,.$$

5. [15] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose that f' is increasing over a bounded interval (a, b). Prove that f is strictly convex over $[a, b]$.

6. [15] Prove that for every nonzero $x \in \mathbb{R}$ we have
   $$1 + \tfrac{6}{5}x < (1 + x)^{\frac{6}{5}}\,.$$

7. [10] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Suppose the equation $f'(x) = 0$ has at most three real solutions. Prove that the equation $f(x) = 0$ has at most four real solutions.

8. [10] Let $D \subset \mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be Hölder continuous of order $\alpha \in (0, 1]$ if there exists a $C \in \mathbb{R}_+$ such that for every $x, y \in D$ we have
   $$|f(x) - f(y)| \le C\,|x - y|^{\alpha}\,.$$
   Show that if $f : D \to \mathbb{R}$ is Hölder continuous of order α for some $\alpha \in (0, 1]$ then it is uniformly continuous over D.