First In-Class Exam Solutions
Math 410, Professor David Levermore
Monday, 1 October 2018

1. [10] Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$.
Write the negations of the following assertions.
   (a) [5] "For every $M > 0$ we have $b_j > M$ eventually as $j \to \infty$."
   (b) [5] "Every sequence in A has a subsequence that converges to a limit in A."

   Solution (a). "There exists an $M > 0$ such that $b_j \leq M$ frequently as $j \to \infty$." □

   Solution (b). "There exists a sequence in A such that every subsequence of it either diverges or converges to a limit outside A." □

   Remark. The answer "There exists a sequence in A such that no subsequence of it converges to a limit in A." does not fully carry the negation through.

   Remark. Assertion (a) is equivalent to the sequence $\{b_k\}$ diverges to ∞. Assertion (b) is the definition that the set A is sequentially compact.

2. [15] Give a counterexample to each of the following false assertions.
   (a) [5] If a sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ is bounded then it converges.
   (b) [5] If $\liminf_{k\to\infty} \dfrac{|b_{k+1}|}{|b_k|} \geq 1$ then $\sum_{k=1}^{\infty} b_k$ diverges.
   (c) [5] A countable union of closed subsets of $\mathbb{R}$ is closed.

   Solution (a). A simple counterexample is the sequence $\{a_k\}_{k\in\mathbb{N}}$ with $a_k = (-1)^k$. It is bounded, but does not converge.

   Solution (b). A simple counterexample is the sequence $\{b_k\}_{k\in\mathbb{N}}$ with $b_k = k^{-2}$. This sequence satisfies

$$\liminf_{k\to\infty} \frac{|b_{k+1}|}{|b_k|} = \liminf_{k\to\infty} \frac{k^2}{(k+1)^2} = \lim_{k\to\infty} \left(\frac{1}{1+\frac{1}{k}}\right)^2 = 1\,,$$

   but the series

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{converges}\,,$$

   because it is the p-series with $p = 2$.

   Remark. This problem is asking for a convergent series about which the Ratio Test is inconclusive. There are many such examples! Any p-series with $p > 1$ is one.

   Solution (c). A simple counterexample is the countable collection of closed intervals given by $[0, 1 - 2^{-n}]$ for every $n \in \mathbb{N}$. Each $[0, 1 - 2^{-n}]$ is closed but their union

$$\bigcup_{n\in\mathbb{N}} [0, 1 - 2^{-n}] = [0, 1) \quad \text{is not closed}\,.$$

3. [10] Consider the real sequence $\{c_k\}_{k\in\mathbb{N}}$ given by

$$c_k = (-1)^k \, \frac{2k-3}{k+1} \qquad \text{for every } k \in \mathbb{N} = \{0, 1, 2, \dots\}\,.$$

(a) [3] Write down the first three terms of the subsequence $\{c_{2k}\}_{k\in\mathbb{N}}$.
(b) [3] Write down the first three terms of the subsequence $\{c_{2k+1}\}_{k\in\mathbb{N}}$.
(c) [4] Write down $\liminf_{k\to\infty} c_k$ and $\limsup_{k\to\infty} c_k$. (No proof is needed here.)

Solution (a). When $k = 0, 1, 2$ we have $2k = 0, 2, 4$, whereby the first three terms of the subsequence $\{c_{2k}\}_{k\in\mathbb{N}}$ are

$$\begin{aligned} c_0 &= (-1)^0 \, \frac{2\cdot 0 - 3}{0+1} = -3\,, \\ c_2 &= (-1)^2 \, \frac{2\cdot 2 - 3}{2+1} = \frac{1}{3}\,, \\ c_4 &= (-1)^4 \, \frac{2\cdot 4 - 3}{4+1} = \frac{5}{5} = 1\,. \end{aligned}$$

Solution (b). When $k = 0, 1, 2$ we have $2k+1 = 1, 3, 5$, whereby the first three terms of the subsequence $\{c_{2k+1}\}_{k\in\mathbb{N}}$ are

$$\begin{aligned} c_1 &= (-1)^1 \, \frac{2\cdot 1 - 3}{1+1} = -\frac{-1}{2} = \frac{1}{2}\,, \\ c_3 &= (-1)^3 \, \frac{2\cdot 3 - 3}{3+1} = -\frac{3}{4}\,, \\ c_5 &= (-1)^5 \, \frac{2\cdot 5 - 3}{5+1} = -\frac{7}{6}\,. \end{aligned}$$

Solution (c). Because $c_{2k+1} < 0$ for $k \geq 1$ while $c_{2k} > 0$ for $k \geq 1$, and because

$$\lim_{k\to\infty} c_{2k+1} = \lim_{k\to\infty} \left((-1)^{2k+1} \frac{2(2k+1)-3}{(2k+1)+1} \right) = -\lim_{k\to\infty} \frac{4k-1}{2k+2} = -2\,,$$

while

$$\lim_{k\to\infty} c_{2k} = \lim_{k\to\infty} \left((-1)^{2k} \frac{2(2k)-3}{(2k)+1} \right) = \lim_{k\to\infty} \frac{4k-3}{2k+1} = 2\,,$$

we see that

$$\liminf_{k\to\infty} c_k = \lim_{k\to\infty} c_{2k+1} = -2\,, \qquad \limsup_{k\to\infty} c_k = \lim_{k\to\infty} c_{2k} = 2\,.$$

4. [15] Let $a_0 > 0$ and define the sequence $\{a_k\}_{k\in\mathbb{N}}$ by $a_{k+1} = \sqrt{a_k + 2}$ for every $k \in \mathbb{N}$.
   (a) [10] Prove that $\{a_k\}_{k\in\mathbb{N}}$ converges.
   (b) [5] Evaluate $\lim_{k\to\infty} a_k$.

Solution (a). Notice that $\{a_k\}_{k\in\mathbb{N}}$ is a positive sequence. We will show that $\{a_k\}_{k\in\mathbb{N}}$ is also a contracting sequence, whereby it will be convergent.

Notice that the recursion relation implies that for every $k \geq 1$ we have

$$\begin{aligned} a_{k+1} - a_k &= \sqrt{a_k + 2} - \sqrt{a_{k-1} + 2} \\ &= \left(\sqrt{a_k + 2} - \sqrt{a_{k-1} + 2}\right) \frac{\sqrt{a_k + 2} + \sqrt{a_{k-1} + 2}}{\sqrt{a_k + 2} + \sqrt{a_{k-1} + 2}} \\ &= \frac{a_k - a_{k-1}}{\sqrt{a_k + 2} + \sqrt{a_{k-1} + 2}} \,. \end{aligned}$$

Therefore, because $\{a_k\}_{k\in\mathbb{N}}$ is a positive sequence, we have for every $k \geq 1$

$$|a_{k+1} - a_k| = \frac{|a_k - a_{k-1}|}{\sqrt{a_k + 2} + \sqrt{a_{k-1} + 2}} < \frac{1}{2\sqrt{2}} \, |a_k - a_{k-1}| \,.$$

Because $1/(2\sqrt{2}) < 1$, this implies that $\{a_k\}_{k\in\mathbb{N}}$ is a contracting sequence, whereby it is convergent. $\square$

Solution (b). Let $a \in \mathbb{R}$ be the limit of the convergent sequence $\{a_k\}_{k\in\mathbb{N}}$. By the recursion relation we have

$$a_{k+1}^2 = a_k + 2 \,.$$

By letting $k \to \infty$ in this relation we see by the properties of limits that

$$a^2 = a + 2 \,,$$

whereby either $a = 2$ or $a = -1$. Because $\{a_k\}_{k\in\mathbb{N}}$ is a positive sequence, we have

$$\lim_{k\to\infty} a_k = 2 \,.$$

$\square$

5. [10] Let A and B be any subsets of $\mathbb{R}$. Prove that $(A \cap B)^c \subset A^c \cap B^c$.
   (Here S^c denotes the closure of any $S \subset \mathbb{R}$.)

Remark. We must show that every element of $(A \cap B)^c$ is an element of $A^c \cap B^c$.

Solution. Let $x \in (A \cap B)^c$ (be arbitrary).

By the definition of closure there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ contained within $A \cap B$ such that $x_n \to x$ as $n \to \infty$.

Because $\{x_n\}_{n\in\mathbb{N}}$ is contained within A and $x_n \to x$ as $n \to \infty$, we see that $x \in A^c$ by the definition of closure.

Because $\{x_n\}_{n\in\mathbb{N}}$ is contained within B and $x_n \to x$ as $n \to \infty$, we see that $x \in B^c$ by the definition of closure.

Because $x \in A^c$ and $x \in B^c$, we know that $x \in A^c \cap B^c$.

Because $x \in (A \cap B)^c$ was arbitrary, we conclude that $(A \cap B)^c \subset A^c \cap B^c$. $\square$

6. [15] Let $\{c_k\}_{k\in\mathbb{N}}$ be a positive sequence in $\mathbb{R}$.
   (a) [10] Prove that

$$\limsup_{k\to\infty} \sqrt[k]{c_k} \leq \limsup_{k\to\infty} \frac{c_{k+1}}{c_k} \,.$$

   (b) [5] Give an example for which the above inequality is strict.

Solution (a). There is nothing to prove when

$$\limsup_{k\to\infty} \frac{c_{k+1}}{c_k} = \infty \,,$$

so suppose that

$$\rho = \limsup_{k\to\infty} \frac{c_{k+1}}{c_k} < \infty \,.$$

Because $\{c_k\}_{k\in\mathbb{N}}$ is a positive sequence, so is $\{c_{k+1}/c_k\}_{k\in\mathbb{N}}$, whereby $\rho \geq 0$. Let $r > \rho$. By Proposition 2.17 we have

$$\frac{c_{k+1}}{c_k} < r \quad \text{eventually} \,,$$

say

$$\frac{c_{k+1}}{c_k} < r \quad \text{for every } k \geq m \,.$$

Because $c_k > 0$ for every $k \in \mathbb{N}$ and $r > \rho \geq 0$, it follows by induction that

$$c_k \leq c_m r^{k-m} \quad \text{for every } k \geq m \,.$$

Therefore

$$\sqrt[k]{c_k} \leq r \sqrt[k]{c_m r^{-m}} \quad \text{for every } k \geq m \,,$$

which implies that

$$\limsup_{k\to\infty} \sqrt[k]{c_k} \leq r \limsup_{k\to\infty} \sqrt[k]{c_m r^{-m}} \leq r \lim_{k\to\infty} \sqrt[k]{c_m r^{-m}} = r \,.$$

Because $r > \rho$ was arbitrary, we have

$$\limsup_{k\to\infty} \sqrt[k]{c_k} \leq \rho = \limsup_{k\to\infty} \frac{c_{k+1}}{c_k} \,.$$

Therefore we have proved the desired inequality. □

Solution (b). One example is given by

$$c_k = \big(3 - (-1)^k\big)^{-k} = \begin{cases} 4^{-k} & \text{for } k \text{ even} \,, \\ 2^{-k} & \text{for } k \text{ odd} \,. \end{cases}$$

It is clear that

$$\limsup_{k\to\infty} \sqrt[k]{c_k} = \lim_{k\to\infty} \sqrt[2k+1]{c_{2k+1}} = \tfrac{1}{2} \,,$$

while

$$\limsup_{k\to\infty} \frac{c_{k+1}}{c_k} = \lim_{k\to\infty} \frac{c_{2k+1}}{c_{2k}} = \lim_{k\to\infty} \frac{4^{2k}}{2^{2k+1}} = \infty \,.$$

Because $\frac{1}{2} < \infty$, the inequality is strict. □

7. [10] Let $\{b_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ be a sequence and $\{b_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of it. Show that

$$\sum_{k=0}^{\infty} b_k \quad \text{converges absolutely} \quad \implies \quad \sum_{k=0}^{\infty} b_{n_k} \quad \text{converges absolutely}\,.$$

Solution. By the definition of absolute convergence of a series

$$\sum_{k=0}^{\infty} b_k \quad \text{converges absolutely} \quad \iff \quad \sum_{k=0}^{\infty} |b_k| \quad \text{converges}\,,$$
$$\sum_{k=0}^{\infty} b_{n_k} \quad \text{converges absolutely} \quad \iff \quad \sum_{k=0}^{\infty} |b_{n_k}| \quad \text{converges}\,.$$

By the definition of a convergent series, each of the series on the right-hand side above is convergent if and only if its associated sequence of partial sums is convergent. These sequences of partial sums are given by $\{q_n\}$ and $\{p_m\}$ respectively where q_n and p_m are defined for every $n, m \in \mathbb{N}$ by

$$q_n = \sum_{k=0}^{n} |b_k|\,, \qquad\qquad p_m = \sum_{k=0}^{m} |b_{n_k}|\,.$$

It is clear that these sequences are nondecreasing. The Monotonic Sequence Theorem then implies that these sequences converge if and only if they are bounded above. Therefore

$$\sum_{k=0}^{\infty} |b_k| \quad \text{converges} \quad \iff \quad \{q_n\} \text{ is bounded above}\,,$$
$$\sum_{k=0}^{\infty} |b_{n_k}| \quad \text{converges} \quad \iff \quad \{p_m\} \text{ is bounded above}\,.$$

The crucial observation is that p_m and q_n satisfy the inequality

$$p_m = \sum_{k=0}^{m} |b_{n_k}| \le \sum_{k=0}^{n_m} |b_k| = q_{n_m} \quad \text{for every } m \in \mathbb{N}\,.$$

This inequality shows that if $\{q_n\}$ is bounded above then $\{p_m\}$ is bounded above. Therefore

$$\begin{aligned} \sum_{k=0}^{\infty} b_k \quad \text{converges absolutely} \quad &\iff \quad \{q_n\} \text{ is bounded above} \\ &\implies \quad \{p_m\} \text{ is bounded above} \\ &\iff \quad \sum_{k=0}^{\infty} b_{n_k} \quad \text{converges absolutely}\,. \end{aligned}$$

□

Remark. This proof involves three notions of convergence: (1) absolute convergence of a series, (2) convergence of a series, and (3) convergence of a sequence. Whenever "converges" appears in your solution it should be clear which notion is being used.

8. [15] Determine the set of all $x \in \mathbb{R}$ for which
$$\sum_{k=0}^{\infty}(-1)^k \frac{3^k x^k}{\sqrt{k+1}} \quad \text{converges}\,.$$
Give your reasoning. (The set is an interval. Be sure to check its endpoints!)

Solution. Let a_k denote the k^{th} term in the sum, namely let
$$a_k = (-1)^k \frac{3^k x^k}{\sqrt{k+1}}\,.$$
We have
$$\limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \limsup_{k\to\infty} \frac{3^{k+1}|x|^{k+1}}{\sqrt{k+2}} \frac{\sqrt{k+1}}{3^k|x|^k} = 3|x| \lim_{k\to\infty} \sqrt{\frac{k+1}{k+2}} = 3|x|\,.$$
The *Ratio Test* shows that the series *converges absolutely* for $3|x| < 1$ and *diverges* for $3|x| > 1$. This test says nothing when $3|x| = 1$.

When $3x = 1$ the series becomes
$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{\sqrt{k+1}}\,.$$
Because the terms $1/\sqrt{k+1}$ are positive and decreasing with
$$\lim_{k\to\infty} \frac{1}{\sqrt{k+1}} = 0\,,$$
the *Alternating Series Test* can be applied to show that the series *converges.*

When $3x = -1$ the series becomes
$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}}\,.$$
Because this is the p-series with $p = \frac{1}{2}$, it *diverges.* Another argument is that because the *harmonic series*
$$\sum_{k=0}^{\infty} \frac{1}{k+1} \quad \text{diverges}\,.$$
and because
$$\frac{1}{k+1} \le \frac{1}{\sqrt{k+1}} \quad \text{for every } k \in \mathbb{N}\,,$$
the *Direct Comparison Test* shows that the series *diverges.* Alternatively, because the terms $1/\sqrt{k+1}$ are positive and decreasing, the *Integral Test* or the *Cauchy* 2^k *Test* can be applied to show that the series *diverges.*

Therefore the set of all $x \in \mathbb{R}$ for which the series converges is the interval
$$\left(-\tfrac{1}{3}\,,\, \tfrac{1}{3}\right]\,.$$
$\square$

Remark. It is not enough to argue that the series converges in the interval $(-\frac{1}{3}, \frac{1}{3}]$. You also have to argue that it diverges outside the interval.