First In-Class Exam
Math 410, Professor David Levermore
Monday, 1 October 2018

1. [10] Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and let A be a subset of $\mathbb{R}$.
Write the negations of the following assertions.
(a) [5] "For every $M > 0$ we have $b_j > M$ eventually as $j \to \infty$."
(b) [5] "Every sequence in A has a subsequence that converges to a limit in A."

2. [15] Give a counterexample to each of the following false assertions.
(a) [5] If a sequence $\{a_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}$ is bounded then it converges.
(b) [5] If $\liminf_{k\to\infty} \dfrac{|b_{k+1}|}{|b_k|} \geq 1$ then $\sum_{k=1}^{\infty} b_k$ diverges.
(c) [5] A countable union of closed subsets of $\mathbb{R}$ is closed.

3. [10] Consider the real sequence $\{c_k\}_{k\in\mathbb{N}}$ given by

$$c_k = (-1)^k \, \frac{2k-3}{k+1} \qquad \text{for every } k \in \mathbb{N} = \{0, 1, 2, \dots\}\,.$$

(a) [3] Write down the first three terms of the subsequence $\{c_{2k}\}_{k\in\mathbb{N}}$.
(b) [3] Write down the first three terms of the subsequence $\{c_{2k+1}\}_{k\in\mathbb{N}}$.
(c) [4] Write down $\liminf_{k\to\infty} c_k$ and $\limsup_{k\to\infty} c_k$. (No proof is needed here.)

4. [15] Let $a_0 > 0$ and define the sequence $\{a_k\}_{k\in\mathbb{N}}$ by $a_{k+1} = \sqrt{a_k + 2}$ for every $k \in \mathbb{N}$.
(a) [10] Prove that $\{a_k\}_{k\in\mathbb{N}}$ converges.
(b) [5] Evaluate $\lim_{k\to\infty} a_k$.

5. [10] Let A and B be any subsets of $\mathbb{R}$. Prove that $(A \cap B)^c \subset A^c \cap B^c$.
(Here S^c denotes the closure of any $S \subset \mathbb{R}$.)

6. [15] Let $\{c_k\}_{k\in\mathbb{N}}$ be a positive sequence in $\mathbb{R}$.
(a) [10] Prove that

$$\limsup_{k\to\infty} \sqrt[k]{c_k} \leq \limsup_{k\to\infty} \frac{c_{k+1}}{c_k}\,.$$

(b) [5] Give an example for which the above inequality is strict.

7. [10] Let $\{b_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ be a sequence and $\{b_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of it. Show that

$$\sum_{k=0}^{\infty} b_k \quad \text{converges absolutely} \qquad \Longrightarrow \qquad \sum_{k=0}^{\infty} b_{n_k} \quad \text{converges absolutely}\,.$$

8. [15] Determine the set of all $x \in \mathbb{R}$ for which

$$\sum_{k=0}^{\infty} (-1)^k \frac{3^k x^k}{\sqrt{k+1}} \quad \text{converges}\,.$$

Give your reasoning. (The set is an interval. Be sure to check its endpoints!)