Quiz 11 Solutions, Math 246, Professor David Levermore
Tuesday, 4 December 2018

(1) [5] Consider the system

$$x' = -2x + y\,, \qquad y' = 5x + 2y - 3x^2\,.$$

(a) [2] Find all of its stationary points.
(b) [3] Find a nonconstant function $H(x,y)$ such that every orbit of this system satisfies $H(x,y) = c$ for some constant c.

Solution (a). The stationary points satisfy

$$0 = -2x + y\,, \qquad 0 = 5x + 2y - 3x^2\,.$$

The first equation is satisfied if and only if $y = 2x$, whereby the second equation becomes $0 = 9x - 3x^2$, which is solved by $x = 0$ or $x = 3$. Therefore all the stationary points are $(0,0)$ and $(3,6)$.

Solution (b). The system is *Hamiltonian* because

$$\partial_x(-2x + y) + \partial_y\big(5x + 2y - 3x^2\big) = -2 + 2 = 0\,,$$

where by the orbit equation is *exact*. Therefore there exists $H(x,y)$ such that

$$\partial_y H(x,y) = -2x + y\,, \qquad -\partial_x H(x,y) = 5x + 2y - 3x^2\,.$$

By integrating the first equation we find that

$$H(x,y) = -2xy + \tfrac{1}{2}y^2 + h(x)\,.$$

By substituting this into the second equation we see that

$$2y - h'(x) = 5x + 2y - 3x^2\,,$$

whereby $h'(x) = -5x + 3x^2$. Therefore we can set

$$H(x,y) = -2xy + \tfrac{1}{2}y^2 - \tfrac{5}{2}x^2 + x^3\,.$$

Group Work Exercises based on Problem 1 [3]

(a) Classify the type and stability of each stationary point.
(b) Sketch a phase-plane portrait of the system that shows its behavior near each stationary point. (Carefully mark all sketched orbits with arrows!)
(c) Add to the phase-plane portrait a sketch of the level set $H(x,y) = c$ for each value of c that corresponds to a stationary point that is a saddle. (Carefully mark all sketched orbits with arrows!)

(2) [5] Consider the system

$$p' = 3p - q\,, \qquad q' = 5p + 5q - 10p^2\,.$$

Its stationary points are $(0,0)$ and $(2,6)$. Classify the type and stability of each of these stationary points. (You do not have to sketch anything.)

Solution. The Jacobian matrix is $\boldsymbol{\partial}\mathbf{f}(p,q) = \begin{pmatrix} \partial_p f & \partial_q f \\ \partial_p g & \partial_q g \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 5 - 20p & 5 \end{pmatrix}$.

- At $(0,0)$ the coefficient matrix of its linearization is $\mathbf{A} = \boldsymbol{\partial}\mathbf{f}(0,0) = \begin{pmatrix} 3 & -1 \\ 5 & 5 \end{pmatrix}$, which has characterisitc polynomial
$$p(\zeta) = \zeta^2 - 8\zeta + 20 = (\zeta + 4)^2 + 2^2\,.$$
The eigenvalues of $\mathbf{A}$ are $4 \pm i2$. Because these eigenvalues are a conjugate pair with positive real part, and because $a_{21} = 5 > 0$, the stationary point $(0,0)$ is a *counterclockwise spiral source* and thereby is *repelling*.
- At $(2,6)$ the coefficient matrix of its linearization is $\mathbf{B} = \boldsymbol{\partial}\mathbf{f}(2,6) = \begin{pmatrix} 3 & -1 \\ -35 & 5 \end{pmatrix}$, which has characteristic polynomial
$$p(\zeta) = \zeta^2 - 8\zeta - 20 = (\zeta - 10)(\zeta + 2)\,.$$
The eigenvalues of $\mathbf{B}$ are 10 and -2. Because these are real, nonzero, and have opposite sign, the stationary point $(2,6)$ is a *saddle* and thereby is *unstable*, but not repelling.

Group Work Exercises based on Problem 2 [3]

(a) Does this system have an integral $H(p,q)$ that is defined over the entire phase-plane? (Either find an integral or give a reason why one does not exist.)

(b) Sketch the phase-plane portrait of the system near the stationary point $(0,0)$. (Carefully mark all sketched orbits with arrows!)

(c) Sketch the phase-plane portrait of the system near the stationary point $(2,6)$. (Carefully mark all sketched orbits with arrows!)

Final Group Work Exercises [4]

Consider the system

$$u' = (15 - 3u - v)u\,, \qquad v' = (24 - 3u - 4v)v\,.$$

Build up a sketch of the phase-plane portrait of this system as follows.

(1) Sketch all stationary points of this system.

(2) Sketch all semistationary orbits of this system.

(3) Classify the type and stability of each stationary point.

(4) Sketch the phase-plane portrait near each stationary point. (Carefully mark all sketched orbits with arrows!)