Quiz 9 Solutions, Math 246, Professor David Levermore
Tuesday, 13 November 2018

(1) [5] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ e^t \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ e^t \end{pmatrix}$.

(a) [2] Compute their Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) [3] Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.

Solution (a). The Wronskian is

$$\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 1 & t^2 \\ e^t & e^t \end{pmatrix} = 1 \cdot e^t - e^t \cdot t^2 = e^t(1 - t^2)\,.$$

Solution (b). Let $\boldsymbol{\Psi}(t) = \begin{pmatrix} 1 & t^2 \\ e^t & e^t \end{pmatrix}$. Because $\boldsymbol{\Psi}'(t) = \mathbf{A}(t)\boldsymbol{\Psi}(t)$, we have

$$\begin{aligned} \mathbf{A}(t) = \boldsymbol{\Psi}'(t)\boldsymbol{\Psi}(t)^{-1} &= \begin{pmatrix} 0 & 2t \\ e^t & e^t \end{pmatrix} \begin{pmatrix} 1 & t^2 \\ e^t & e^t \end{pmatrix}^{-1} \\ &= \frac{1}{e^t(1 - t^2)} \begin{pmatrix} 0 & 2t \\ e^t & e^t \end{pmatrix} \begin{pmatrix} e^t & -t^2 \\ -e^t & 1 \end{pmatrix} = \frac{1}{e^t(1 - t^2)} \begin{pmatrix} -2te^t & 2t \\ 0 & (1 - t^2)e^t \end{pmatrix}. \end{aligned}$$

Remark. Group Work Exercises related to this problem were done last week.

(2) [4] Let $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}$. Compute $e^{t\mathbf{A}}$.

Solution. Because $\mathbf{A}$ is 2×2 its characteristic polynomial is

$$p(\zeta) = \zeta^2 - \mathrm{tr}(\mathbf{A})\zeta + \det(\mathbf{A}) = \zeta^2 - 4\zeta + 13 = (\zeta - 2)^2 + 3^2\,.$$

Because this is a *sum of squares* with roots $2 \pm i3$, we have

$$\begin{aligned} e^{t\mathbf{A}} &= e^{2t}\left[\cos(3t)\mathbf{I} + \frac{\sin(3t)}{3}(\mathbf{A} - 2\mathbf{I})\right] \\ &= e^{2t}\left[\cos(3t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin(3t)}{3}\begin{pmatrix} -1 & -2 \\ 5 & 1 \end{pmatrix}\right] \\ &= e^{2t}\begin{pmatrix} \cos(3t) - \frac{1}{3}\sin(3t) & -\frac{2}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) & \cos(3t) + \frac{1}{3}\sin(3t) \end{pmatrix}. \end{aligned}$$

Group Work Exercises for Problem 2 [4]

(a) Give a general solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(b) Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(c) Give the natural fundamental matrix for $t_I = 3$ to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(d) Give an eigenpair for each eigenvalue of $\mathbf{A}$.

(3) [1] Suppose that $e^{t\mathbf{A}} = e^{2t}\begin{pmatrix} \cosh(3t) & \frac{1}{3}\sinh(3t) \\ 3\sinh(3t) & \cosh(3t) \end{pmatrix}$.

Compute the Green matrix $\mathbf{G}(t,s)$ associated with $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution. The Green matrix is

$$\mathbf{G}(t,s) = e^{(t-s)\mathbf{A}} = e^{2(t-s)}\begin{pmatrix} \cosh(3(t-s)) & \frac{1}{3}\sinh(3(t-s)) \\ 3\sinh(3(t-s)) & \cosh(3(t-s)) \end{pmatrix}.$$

Group Work Exercises for Problem 3 [2]

(a) Find $\mathbf{A}$.

(b) Solve the initial-value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

Group Work Exercises for Exam 3 [4]

Consider the matrix $\mathbf{C} = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$.

(1) Find the eigenvalues of $\mathbf{C}$.

(2) Give an eigenpair for each eigenvalue of $\mathbf{C}$.

(3) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$.

(4) Use $\mathbf{V}$ and $\mathbf{D}$ to compute $e^{t\mathbf{C}}$.