Quiz 8 Solutions, Math 246, Professor David Levermore
Tuesday, 6 November 2018

Short Table: $\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$ for $s > a$, $\qquad \mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$.

(1) [5] Find $F(s) = \mathcal{L}[f](s)$ where $f(t) = u(t-3)e^{-2t} + 4\delta(t-5)$.

Solution. By linearity we have

$$\mathcal{L}[f](s) = \mathcal{L}[u(t-3)e^{-2t}](s) + 4\mathcal{L}[\delta(t-5)](s).$$

By the shifty step method $u(t-3)e^{-2t} = u(t-3)j(t-3)$ where

$$j(t) = e^{-2(t+3)} = e^{-2t}e^{-6}.$$

Hence, the second entry in our short table with $c = 3$ gives

$$\mathcal{L}[u(t-3)e^{-2t}](s) = \mathcal{L}[u(t-3)j(t-3)](s) = e^{-3s}\mathcal{L}[j](s) = e^{-3s}e^{-6}\mathcal{L}[e^{-2t}](s).$$

The first entry in our short table with $n = 0$ and $a = -2$ gives

$$\mathcal{L}[e^{-2t}](s) = \frac{1}{s+2}.$$

Therefore

$$\mathcal{L}[u(t-3)e^{-2t}](s) = e^{-3s}e^{-6}\frac{1}{s+2}.$$

By the rule for evaluating the unit impulse under an integral we have

$$\mathcal{L}[\delta(t-5)](s) = \int_0^\infty e^{-st}\delta(t-5)\,\mathrm{d}t = e^{-5s}.$$

Therefore

$$\mathcal{L}[f](s) = \frac{e^{-3s-6}}{s+2} + 4e^{-5s}.$$

(2) [2] Transform the equation $v'''' - e^{v'}v'' - \sin(t+v) = 0$ into a first-order system of ordinary differential equations.

Solution. Because the equation is fourth order, the first-order system must have dimension four. The simplest such first-order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ e^{x_2}x_3 + \sin(t+x_1) \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} v \\ v' \\ v'' \\ v''' \end{pmatrix}.$$

Remark. There should be no v, v', v'', or v''' appearing in the first-order system. The only place these should appear is in the dictionary on the right that shows their relationship to the new variables. The first-order system should be expressed solely in terms of the new variables. The new variables are x_1, x_2, x_3, and x_4 in the solution above. Any similar set of new variables could be used.

Group Work Exercises for Problem 2 [3]

(a) Two interconnected tanks contain brine (salt water). At $t = 0$ the first tank contains 23 liters and the second contains 32 liters. Brine with a salt concentration of 8 grams per liter flows into the first tank at 6 liters per hour. Well-stirred brine flows from the first tank into the second at 7 liters per hour, from the second into the first at 5 liters per hour, from the first into a drain at 3 liter per hour, and from the second into a drain at 4 liters per hour. At $t = 0$ there are 17 grams of salt in the first tank and 29 grams in the second. Give an initial-value problem that governs the grams of salt in each tank as a function of time.

(b) Give the interval of definition for the solution of the above initial-value problem.

(c) Express the above initial-value problem in the form

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)\,, \qquad \mathbf{x}(0) = \mathbf{x}^I\,.$$

Give $\mathbf{A}(t)$, $\mathbf{f}(t)$, and $\mathbf{x}^I$.

(3) [3] Consider the matrix-valued function $\boldsymbol{\Psi}(t) = \begin{pmatrix} 1 & -2t^2 \\ t^2 & 4 - t^4 \end{pmatrix}$.

(a) Compute $\det\big(\boldsymbol{\Psi}(t)\big)$.
(b) Compute $\boldsymbol{\Psi}(t)^{-1}$.
(c) Compute $\boldsymbol{\Psi}'(t)$.

Solution (a). The determinant of $\boldsymbol{\Psi}(t)$ is

$$\det\big(\boldsymbol{\Psi}(t)\big) = \det\begin{pmatrix} 1 & -2t^2 \\ t^2 & 4 - t^4 \end{pmatrix} = 1 \cdot (4 - t^4) - t^2 \cdot (-2t^2) = 4 + t^4\,.$$

Solution (b). The inverse of $\boldsymbol{\Psi}(t)$ is

$$\boldsymbol{\Psi}(t)^{-1} = \frac{1}{\det\big(\boldsymbol{\Psi}(t)\big)} \begin{pmatrix} 4 - t^4 & 2t^2 \\ -t^2 & 1 \end{pmatrix} = \frac{1}{4 + t^4} \begin{pmatrix} 4 - t^4 & 2t^2 \\ -t^2 & 1 \end{pmatrix}\,.$$

Solution (c). The derivative of $\boldsymbol{\Psi}(t)$ is

$$\boldsymbol{\Psi}'(t) = \begin{pmatrix} 0 & -4t \\ 2t & -4t^3 \end{pmatrix}\,.$$

Group Work Exercises for Problem 3 [7]

Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} -2t^2 \\ 4 - t^4 \end{pmatrix}$.

Some of the solutions to Problem 3 can be helpful.

(a) Compute their Wronskian $\mathrm{Wr}[\mathbf{x}_1, \mathbf{x}_2](t)$.
(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$.
(c) Give a general solution of this system.
(d) Give a fundamental matrix for this system.
(e) Find the natural fundamental matrix of this system for $t_I = 0$.
(f) Find the natural fundamental matrix of this system for $t_I = 2$.
(g) Find the Green matrix $\mathbf{G}(t, s)$ associated with this system.