Quiz 7 Solutions, Math 246, Professor David Levermore
Tuesday, 30 October 2018

Short Table: $\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$ for $s > a$, $\quad \mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)$.

(1) [4] Use the definition of the Laplace transform to compute $\mathcal{L}[f](s)$ for the function $f(t) = u(t-2)e^{3t}$, where u is the unit step function.

Solution. By the definitions of the Laplace transform and the unit step function

$$\begin{aligned}\mathcal{L}[f](s) &= \lim_{T\to\infty}\int_0^T e^{-st}f(t)\,\mathrm{d}t = \lim_{T\to\infty}\int_0^T e^{-st}u(t-2)e^{3t}\,\mathrm{d}t \\ &= \lim_{T\to\infty}\int_2^T e^{-st}e^{3t}\,\mathrm{d}t = \lim_{T\to\infty}\int_2^T e^{-(s-3)t}\,\mathrm{d}t\,.\end{aligned}$$

For $s \leq 3$ we have $e^{-(s-3)t} \geq 1$, so for $T > 2$

$$\int_2^T e^{-(s-3)t}\,\mathrm{d}t \geq \int_2^T \mathrm{d}t = T - 2\,,$$

whereby $\mathcal{L}[f](s)$ is undefined for $s \leq 3$ because

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\int_2^T e^{-(s-3)t}\,\mathrm{d}t \geq \lim_{T\to\infty}(T-2) = \infty \qquad \text{for } s \leq 3\,.$$

For $s > 3$ and $T > 2$

$$\int_2^T e^{-(s-3)t}\,\mathrm{d}t = -\frac{e^{-(s-3)t}}{s-3}\bigg|_2^T = \frac{e^{-(s-3)2}}{s-3} - \frac{e^{-(s-3)T}}{s-3}\,,$$

whereby

$$\begin{aligned}\mathcal{L}[f](s) &= \lim_{T\to\infty}\int_2^T e^{-(s-3)t}\,\mathrm{d}t \\ &= \lim_{T\to\infty}\left[\frac{e^{-(s-3)2}}{s-3} - \frac{e^{-(s-3)T}}{s-3}\right] = \frac{e^{-(s-3)2}}{s-3} \qquad \text{for } s > 3\,.\end{aligned}$$

(2) [3] Find the Laplace transform $X(s)$ of the solution $x(t)$ of the initial-value problem

$$x'' - 9x = 0\,, \qquad x(0) = 2\,, \quad x'(0) = -4\,.$$

DO NOT solve for $x(t)$, just $X(s)$!

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[x''](s) - 9\mathcal{L}[x](s) = 0\,,$$

where

$$\begin{aligned}\mathcal{L}[x](s) &= X(s)\,, \\ \mathcal{L}[x'](s) &= s\mathcal{L}[x](s) - x(0) = sX(s) - 2\,, \\ \mathcal{L}[x''](s) &= s\mathcal{L}[x'](s) - x'(0) = s\big(sX(s) - 2\big) + 4 = s^2X(s) - 2s + 4\,.\end{aligned}$$

By placing these into the Laplace transform of the initial-value problem gives

$$\big(s^2X(s) - 2s + 4\big) - 9X(s) = 0\,,$$

which yields

$$(s^2 - 9)X(s) - 2s + 4 = 0\,,$$

whereby

$$X(s) = \frac{2s-4}{s^2-9}\,.$$

Group Work Exercises on Problems 1 and 2 [5]

Let $f(t)$ be as in Problem 1 and $x(t)$ be as in Problem 2.

Short Table: $\mathcal{L}[t^n e^{at}](s) = \dfrac{n!}{(s-a)^{n+1}}$ for $s > a\,,$ $\quad \mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}\mathcal{L}[j](s)\,.$

(a) Use the short table to compute $\mathcal{L}[f](s)$.

(b) Use the short table to compute $x(t) = \mathcal{L}^{-1}[X](t)$.

(c) Use the Laplace transform to solve the initial-value problem

$$v'' - 9v = f(t)\,, \qquad v(0) = 0\,, \quad v'(0) = 0\,.$$

(d) Use the Laplace transform to compute the Green function $g(t)$ for the operator $\mathrm{D}^2 - 9$.

(e) Compute the natural fundamental set of solutions $N_0(t)$, $N_1(t)$ associated with initial time 0 for the operator $\mathrm{D}^2 - 9$.

(3) [3] Find $y(t) = \mathcal{L}^{-1}[Y](t)$ where $Y(s) = e^{-4s}\dfrac{15}{(s-2)(s+3)}$.

Solution. By the partial fraction identity

$$J(s) = \frac{15}{(s-2)(s+3)} = \frac{3}{s-2} + \frac{-3}{s+3}\,,$$

and by the first entry in our table with $n = 0$ and $a = 2$ and with $n = 0$ and $a = -3$ we have

$$\begin{aligned} j(t) &= \mathcal{L}^{-1}\left[\frac{15}{(s-2)(s+3)}\right](t) \\ &= 3\mathcal{L}^{-1}\left[\frac{1}{s-2}\right](t) - 3\mathcal{L}^{-1}\left[\frac{1}{s+3}\right](t) = 3e^{2t} - 3e^{-3t}\,. \end{aligned}$$

Therefore by the second entry in our table we have

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[Y](t) = \mathcal{L}^{-1}\big[e^{-4s}J(s)\big](t) = u(t-4)j(t-4) \\ &= u(t-4)\big(3e^{2(t-4)} - 3e^{-3(t-4)}\big) = 3u(t-4)\big(e^{2t-8} - e^{-3t+12}\big)\,. \end{aligned}$$

Group Work Exercises for Problems 3 [5]

(a) Find $x(t) = \mathcal{L}^{-1}[X](t)$ where $X(s) = e^{-4s}\dfrac{15}{(s^2-2)(s^2+3)}$.

(b) Find $y(t) = \mathcal{L}^{-1}[Y](t)$ where $Y(s) = e^{-3s}\dfrac{2s+14}{s^2+10s+29}$.

(c) Find $F(s) = \mathcal{L}[f](s)$ where $f(t) = u(t-3)e^{-2t}\cos(5t) + 7\delta(t-4)$.

(d) Find $F(s) = \mathcal{L}[f](s)$ where $f(t) = u(t-3)t^2e^{-2t}\cos(5t)$.

(e) Find $F(s) = \mathcal{L}[f](s)$ where

$$f(t) = \begin{cases} \cos(4t) & \text{for } 0 \le t < \pi\,, \\ 1 & \text{for } \pi \le t < 6\,, \\ e^{6-t} & \text{for } 6 \le t < \infty\,. \end{cases}$$

A Longer Table of Laplace Transforms

$$\mathcal{L}[t^n e^{at}](s) = \frac{n!}{(s-a)^{n+1}} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\cos(bt)](s) = \frac{s-a}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[e^{at}\sin(bt)](s) = \frac{b}{(s-a)^2+b^2} \qquad \text{for } s > a\,.$$

$$\mathcal{L}[t^n j(t)](s) = (-1)^n J^{(n)}(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[e^{at} j(t)](s) = J(s-a) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s)\,.$$

$$\mathcal{L}[u(t-c)j(t-c)](s) = e^{-cs}J(s) \qquad \text{where } J(s) = \mathcal{L}[j(t)](s),\ c \ge 0,$$
and u is the unit step function.

$$\mathcal{L}[\delta(t-c)h(t)](s) = e^{-cs}h(c) \qquad \text{where } c \ge 0$$
and δ is the unit impluse.