Quiz 6 Solutions, Math 246, Professor David Levermore
Tuesday, 16 October 2018

(1) [5] Compute the Green function for the differential operator $\mathrm{L} = \mathrm{D}^2 + 4\mathrm{D} + 29$.

Solution. The Green function $g(t)$ for L solves the initial-value problem

$$g'' + 4g' + 29g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The associated characteristic polynomial is

$$p(\zeta) = \zeta^2 + 4\zeta + 29 = (\zeta + 2)^2 + 5^2\,,$$

which has roots $-2 \pm i5$. Therefore a general solution of the equation is

$$g(t) = c_1 e^{-2t}\cos(5t) + c_2 e^{-2t}\sin(5t)\,.$$

Because $g(0) = c_1$, the initial condition $g(0) = 0$ implies that $c_1 = 0$. Therefore

$$g(t) = c_2 e^{-2t}\sin(5t)\,, \qquad g'(t) = 5c_2 e^{-2t}\cos(5t) - 2c_2 e^{-2t}\sin(5t)\,.$$

Because $g'(0) = 5c_2$, the initial condition $g'(0) = 1$ implies that $c_2 = \frac{1}{5}$. Therefore the Green function for L is

$$g(t) = \tfrac{1}{5} e^{-2t}\sin(5t)\,.$$

Group Work Exercises for Problem 1 [2]

(a) Solve the initial-value problem

$$x'' + 4x' + 29x = \frac{e^{-2t}}{\cos(5t)}\,, \qquad x(0) = 0\,, \quad x'(0) = 0\,.$$

Give the interval of definition of the solution.

(b) Solve the initial-value problem

$$y'' + 4y' + 29y = \frac{e^{-2t}}{\cos(5t)}\,, \qquad y(0) = 1\,, \quad y'(0) = 0\,.$$

Hint: You can use the answer to (a) as the particular solution.

(2) [3] Find the amplitude and phase of the simple harmonic motion

$$h(t) = 5\cos(3t) - 12\sin(3t)\,.$$

Solution. The point in the plane with Cartesian coordinates $(5, -12)$ lies in the fourth quadrant and has polar coordinates (a, ϕ) with

$$a = \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13\,,$$
$$\phi = 2\pi - \tan^{-1}\big(\tfrac{12}{5}\big)\,.$$

Therefore the amplitude is $a = 13$ and the phase is $\phi = 2\pi - \tan^{-1}\big(\frac{12}{5}\big)$.

Remark. There are many ways to express ϕ. For example, because ϕ is in the fourth quadrant we know that $\frac{3\pi}{2} < \phi < 2\pi$. Using either 2π or $\frac{3\pi}{2}$ as a reference we have

$$\begin{aligned} \phi &= 2\pi - \tan^{-1}\!\left(\tfrac{12}{5}\right), \qquad & \phi &= \tfrac{3\pi}{2} + \tan^{-1}\!\left(\tfrac{5}{12}\right), \\ \phi &= 2\pi - \sin^{-1}\!\left(\tfrac{12}{13}\right), \qquad & \phi &= \tfrac{3\pi}{2} + \sin^{-1}\!\left(\tfrac{5}{13}\right), \\ \phi &= 2\pi - \cos^{-1}\!\left(\tfrac{5}{13}\right), \qquad & \phi &= \tfrac{3\pi}{2} + \cos^{-1}\!\left(\tfrac{12}{13}\right). \end{aligned}$$

The first column uses 2π as the reference while the second uses $\frac{3\pi}{2}$. Other inverse trigonometric functions could have been used. Only one correct answer (and no wrong answers) was required for full credit.

Remark. This oscillation has frequency 3 and period $\frac{2\pi}{3}$.

(3) [2] The displacement $h(t)$ of a spring-mass system is governed by

$$\ddot{h} + 2\eta\dot{h} + 25h = f(t),$$

where $\eta \geq 0$ and $f(t)$ is a forcing. For what values of η is the system under damped?

Solution. The system is under damped when $0 < \eta < \omega_o$. Because the natural frequency of this system is $\omega_o = \sqrt{25} = 5$, the system is under damped when

$$0 < \eta < 5.$$

Alternative Solution. The system is under damped when $\eta > 0$ and the associated characteristic polynomial has conjugate roots. Because the associated characteristic polynomial is

$$p(\zeta) = \zeta^2 + 2\eta\zeta + 25 = (\zeta + \eta)^2 + 25 - \eta^2,$$

it has a conjugate pair of roots whenever $25 - \eta^2 > 0$. Therefore the system is under damped when

$$0 < \eta < 5.$$

Group Work Exercises for Problems 2 and 3 [5]

(a) Express $h(t) = 5\cos(3t) - 12\sin(3t)$ in both its Cartesian and polar phasor form.

(b) For what values of $\eta \geq 0$ is the system in Problem 3
  (i) undamped?
  (ii) critically damped?
  (iii) over damped?

(c) Let $\eta = 0$ and $f(t) = 7\cos(\omega t)$. For what value of ω does resonance occur for the system in Problem 3?

(d) Let $\eta = 3$ and $f(t) = 5\cos(3t) - 12\sin(3t)$. Give the steady-state solution for the system in Problem 3 in its Cartesian phasor form.

(e) Let $\eta = 3$ and $f(t) = 0$. Give the solution of the system in Problem 3 that satisfies the initial conditions $h(0) = 0$ and $\dot{h}(0) = 6$. Put it in amplitude-phase form. Give the natural frequency ω_o, natural period T_o, damped frquency ω_η, and damped period T_η of this system.

Group Work Exercises for Exam 2 [3]

The functions $1+t$ and e^t are solutions of the homogeneous equation

$$t\,x'' - (1+t)x' + x = 0 \qquad \text{over } t > 0\,.$$

(You do not have to check that this is true!)

(1) Compute the general Green function $G(t,s)$ associated with $1+t$ and e^t.

(2) Solve the initial-value problem

$$t\,y'' - (1+t)y' + y = -\frac{t^2}{1+t} \qquad y(2) = 0\,, \quad y'(2) = 0\,.$$

(3) Give a general solution of the nonhomogeneous equation

$$t\,y'' - (1+t)y' + y = -\frac{t^2}{1+t} \qquad \text{over } t > 0\,.$$