Quiz 5 Solutions, Math 246, Professor David Levermore
Tuesday, 9 October 2018

(1) [2] Given that $2e^{-t}$ is a particular solution of the equation

$$u'' + 4u' + 13u = 20e^{-t},$$

give a real general solution.

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 + 4\mathrm{D} + 13$. Its characteristic polynomial is $p(z) = z^2 + 4z + 25 = (z+2)^2 + 3^2$, which has conjugate pair of roots $-2 \pm i3$. Therefore a real general solution of the associated homogeneous equation is

$$u_H(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t).$$

Because we are given that a particular solution is $u_P(t) = 2e^{-t}$, a real general solution of the nonhomogeneous equation is

$$u(t) = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) + 2e^{-t}.$$

Remark. The forcing $20e^{-t}$ has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = -1$, and multiplicity $m = 0$. Therefore you should be able to find a particular solution of the equation by using either Key Identity Evaluations, the Zero Degree Formula, Undetermined Coefficients, or the Green Function. Try all four methods! Do they give the same result?

Group Work Exercises for Problem 1 [4]

(a) Find the solution of the differential equation that satisfies the initial conditions $u(0) = 3$, $u'(0) = 5$.

(b) Compute the Green function $g(t)$ for the differential operator $\mathrm{L} = \mathrm{D}^2 + 4\mathrm{D} + 13$.

(c) Use the Green function to express the particular solution of $\mathrm{L}(u) = 20e^{-t}$ that satisfies the initial conditions $u(0) = u'(0) = 0$ in terms of two definite integrals. (Do not evaluate the integrals!)

(d) Solve the initial-value problem

$$x'' + 4x' + 13x = \frac{e^{-2t}}{\cos(3t)}, \qquad x(0) = x'(0) = 0.$$

(Here you should evaluate any integrals!)

(2) [3] Give the degree, characteristic, and multiplicity for the forcing term of the equation

$$v'' + 4v' + 13v = 7t^4 e^{-2t}\cos(3t)\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 + 4\mathrm{D} + 13$. Its characteristic polynomial is $p(z) = z^2 + 4z + 13 = (z+2)^2 + 3^2$, which has conjugate pair of roots $-2 \pm i3$.

The forcing term $7t^4e^{-2t}\cos(3t)$ has degree $d = 4$, characteristic $\mu + i\nu = -2 + i3$, and multiplicity $m = 1$.

Group Work Exercises for Problem 2 [3]

(a) Write down the Key Identity for this equation and all of the derivatives of the Key Identity with respect to z that are needed to find a particular solution of this equation by the method of Key Identity Evaluations. (Do not evaluate these at the characteristic, as you would do to carry out the method!)

(b) If the method of Undetermined Coefficients is used to find a particular solution of this equation, what form will the solution take? (Do not carry out the method!)

(c) Use the Green function to express the particular solution of $\mathrm{L}(v) = 7t^4e^{-2t}\cos(3t)$ that satisfies the initial conditions $v(0) = v'(0) = 0$ in terms of two definite integrals. (Do not evaluate the integrals!)

(3) [5] Find a particular solution of the equation

$$w'' - 6w' + 9w = 8e^{3t}\,.$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its linear operator is $\mathrm{L} = \mathrm{D}^2 - 6\mathrm{D} + 9$. Its characteristic polynomial is $p(z) = z^2 - 6z + 9 = (z-3)^2$, which has the double root 3.

Its forcing has characteristic form with degree $d = 0$, characteristic $\mu + i\nu = 3$, and multiplicity $m = 2$. A particular solution $w_P(t)$ should be found by using either Key Identity Evaluations, the Zero Degree Formula, or Undetermined Coefficients. Below we show that each of these methods gives the particular solution

$$w_P(t) = 4t^2e^{3t}\,.$$

Key Identity Evaluations. Because $d = 0$ and $m = 2$, we need to evaluate the second derivative of the Key Identity at $z = \mu + i\nu = 3$. Because $p(z) = z^2 - 6z + 9$, the Key Identity and its first two derivatives with respect to z are

$$\begin{aligned}
\mathrm{L}(e^{zt}) &= (z^2 - 6z + 9)e^{zt}\,,\\
\mathrm{L}(t\,e^{zt}) &= (z^2 - 6z + 9)t\,e^{zt} + (2z-6)e^{zt}\,,\\
\mathrm{L}(t^2e^{zt}) &= (z^2 - 6z + 9)t^2e^{zt} + 2(2z-6)t\,e^{zt} + 2e^{zt}\,.
\end{aligned}$$

By evaluating the second derivative of the Key Identity at $z = 3$ we obtain

$$\mathrm{L}(t^2e^{3t}) = (3^2 - 6\cdot 3 + 9)t^2e^{3t} + 2(2\cdot 3 - 6)t\,e^{3t} + 2e^{3t} = 2e^{3t}\,.$$

Therefore a particular solution of $\mathrm{L}(w) = 8e^{3t}$ is

$$w_P(t) = 4t^2e^{3t}\,.$$

Zero Degree Formula. This formula can be used because $d = 0$. For a forcing in the form

$$f(t) = \alpha e^{\mu t}\cos(\nu t) + \beta e^{\mu t}\sin(\nu t)\,,$$

it gives the particular solution

$$w_P(t) = t^m e^{\mu t}\,\mathrm{Re}\left(\frac{(\alpha - i\beta)e^{i\nu t}}{p^{(m)}(\mu + i\nu)}\right).$$

For this problem $f(t) = 8e^{3t}$ and $p(z) = z^2 - 6z + 9$, so that $\mu = 3$, $\nu = 0$, $\alpha = 8$, $\beta = 0$, $m = 2$, and $p''(z) = 2$, whereby

$$w_P(t) = t^2 e^{3t}\,\frac{8}{p''(3)} = \frac{8}{2}\,t^2 e^{3t} = 4t^2 e^{3t}\,.$$

Undetermined Coefficients. Because $m + d = 2$, $m = 2$, and $\mu + i\nu = 3$, there is a particular solution of $\mathrm{L}(w) = 8e^{3t}$ in the form

$$w_P(t) = At^2 e^{3t}\,.$$

By taking derivatives we get

$$\begin{aligned}
w_P'(t) &= At^2 \cdot 3e^{3t} + A2t \cdot e^{3t} \\
&= 3At^2 e^{3t} + 2At\,e^{3t}\,, \\
w_P''(t) &= At^2 \cdot 9e^{3t} + 2A2t \cdot 3e^{3t} + A2 \cdot e^{3t} \\
&= 9At^2 e^{3t} + 12At\,e^{3t} + 2A\,e^{3t}\,,
\end{aligned}$$

whereby

$$\begin{aligned}
\mathrm{L}\big(w_P(t)\big) &= w_P''(t) - 6w_P'(t) + 9w_P(t) \\
&= \big[9At^2 e^{3t} + 12At\,e^{3t} + 2A\,e^{3t}\big] - 6\big[3At^2 e^{3t} + 2At\,e^{3t}\big] + 9At^2 e^{3t} \\
&= 2Ae^{3t}\,.
\end{aligned}$$

By setting $2Ae^{3t} = 8e^{3t}$ we see that $2A = 8$, whereby $A = 4$. Therefore a particular solution of $\mathrm{L}(w) = 8e^{3t}$ is

$$w_P(t) = 4t^2 e^{3t}\,.$$

Remark. Because a real general solution of the associated homogeneous equation is

$$w_H(t) = c_1 e^{3t} + c_2 t\,e^{3t}\,,$$

a real general solution of the nonhomogeneous equation $\mathrm{L}(w) = 8e^{3t}$ is

$$w(t) = c_1 e^{3t} + c_2 t\,e^{3t} + 4t^2 e^{3t}\,.$$

Group Work Exercises for Problem 3 [3]

(a) Use the general solution given above to find the solution of $\mathrm{L}(w) = 8e^{3t}$ that satisfies the initial conditions $w(5) = w'(5) = 0$.

(b) Compute the Green function $g(t)$ for the differential operator $\mathrm{L} = \mathrm{D}^2 - 6\mathrm{D} + 9$.

(c) Use the Green Function method to find the particular solution of $\mathrm{L}(w) = 8e^{3t}$ that satisfies the initial conditions $w(5) = w'(5) = 0$.