Quiz 4 Solutions, Math 246, Professor David Levermore
Tuesday, 2 October 2018

(1) [3] Determine the interval of definition for the solution to the initial-value problem

$$u''' + \frac{1}{2-t}\,u'' - \frac{e^t}{\sin(t)}\,u = \frac{\cos(2t)}{8+t}\,, \qquad u(-7) = u'(-7) = u''(-7) = 3\,.$$

Solution. This nonhomogeneous linear equation for u is already in normal form. Notice that

- ◇ the coefficient of u'' is undefined at $t = 2$ and is continuous elsewhere;
- ◇ the coefficient of u is undefined at $t = n\pi$ for every integer n and is continuous elsewhere;
- ◇ the forcing is undefined at $t = -8$ and is continuous elsewhere;
- ◇ the initial time is $t = -7$.

Therefore the interval of definition is $(-8, -2\pi)$ because

- the initial time -7 is in $(-8, -2\pi)$,
- all the coefficients and the forcing are continuous over $(-8, -2\pi)$,
- the forcing is undefined at $t = -8$,
- the coefficient of u is undefined at $t = -2\pi$.

Remark. All four reasons must be given for full credit.

- ○ The first two reasons are why a (unique) solution exists over the interval $(-8, -2\pi)$.
- ○ The last two reasons are why this solution does not exist over a larger interval.

Group Work Exercises for Problem 1 [3]

(a) Suppose that we plan to approximate the solution of the initial-value problem numerically.

(i) Does it make sense or not to do this over the time interval $[-7, -6.1]$?
(ii) Does it make sense or not to do this over the time interval $[-7, -6.2]$?
(iii) Does it make sense or not to do this over the time interval $[-7, -6.3]$?

(b) What is the interval of definition if the initial conditions are

$$u(3) = 0\,, \quad u'(3) = -4\,, \quad u''(3) = -3\,?$$

(c) Suppose that $U_1(t)$, $U_2(t)$, and $U_3(t)$ are solutions to the associated homogeneous differential equation over $(0, 2)$. Suppose that $\mathrm{Wr}[U_1, U_2, U_3](1) = 5$. What is $\mathrm{Wr}[U_1, U_2, U_3](t)$ for every t in $(0, 2)$? Hint: Abel.

(2) [3] Compute the Wronskian $\mathrm{Wr}[V_1, V_2](t)$ of the functions $V_1(t) = e^{3t}$ and $V_2(t) = t\,e^{3t}$. (Evaluate the determinant and simplify.)

Solution. Because $V_1'(t) = 3e^{3t}$ and $V_2'(t) = e^{3t} + 3t\,e^{3t}$, the Wronskian is

$$\mathrm{Wr}[V_1, V_2](t) = \det\begin{pmatrix} V_1(t) & V_2(t) \\ V_1'(t) & V_2'(t) \end{pmatrix} = \det\begin{pmatrix} e^{3t} & t\,e^{3t} \\ 3e^{3t} & e^{3t} + 3t\,e^{3t} \end{pmatrix}$$
$$= e^{3t}\big(e^{3t} + 3t\,e^{3t}\big) - 3e^{3t}\big(t\,e^{3t}\big) = e^{6t} + 3t\,e^{6t} - 3t\,e^{6t} = e^{6t}\,.$$

(3) [4] Given that e^{3t} and $t\,e^{3t}$ are linearly independent solutions of $v'' - 6v' + 9v = 0$, solve the general initial-value problem associated with $t = 0$ — namely, solve

$$v'' - 6v' + 9v = 0\,, \qquad v(0) = v_0\,, \quad v'(0) = v_1\,.$$

Solution. This is a homogeneous linear equation with constant coefficients. Because we are given that e^{3t} and $t\,e^{3t}$ are solutions to it, we can use the method of linear superposition to seek the solution of the general initial-value problem in the form

$$v(t) = c_1 e^{3t} + c_2 t\,e^{3t}\,.$$

Then $v'(t) = c_1 3e^{3t} + c_2(e^{3t} + 3t\,e^{3t})$ and the initial conditions yield

$$v_0 = v(0) = c_1\,, \qquad\qquad v_1 = v'(0) = 3c_1 + c_2\,.$$

It follows that

$$c_1 = v_0\,, \qquad c_2 = v_1 - 3v_0\,.$$

Therefore the solution of the general initial-value problem is

$$v = v_0 e^{3t} + (v_1 - 3v_0)t\,e^{3t}\,.$$

Group Work Exercises for Problems 2 and 3 [4]

Use the solutions of Problems 2 and 3 to help answer the following.

(a) Why are e^{3t} and $t\,e^{3t}$ are linearly independent functions?

(b) Why do e^{3t} and $t\,e^{3t}$ comprise a fundamental set of solutions to this equation? Use them to give a general solution to this equation.

(c) How can we know that $\mathrm{Wr}[V_1, V_2](t)$ is proportional to e^{6t} without computing $\mathrm{Wr}[V_1, V_2](t)$? Hint: Abel.

(d) Find the natural fundamental set of solutions for this equation associated with the initial time $t = 0$.

Group Work Exercises for Quiz 5 [3]

(1) Give a real general solution of the equation

$$x'''' + 4x''' - 5x'' = 0\,.$$

(2) Give a real general solution to the equation

$$(\mathrm{D}^2 - 6\mathrm{D} + 13)^2(\mathrm{D} + 5)^3 y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

(3) Give a real general solution to the equation

$$u'' + 9u = 20e^t\,,$$

given that $2e^t$ is a solution to it and that $\cos(3t)$ and $\sin(3t)$ are a fundamental set of solutions to the associated homogeneous equation.