Quiz 3 Solutions, Math 246, Professor David Levermore
Tuesday, 18 September 2018

(1) [2] Suppose we have used the Runge-Kutta method to approximate the solution of an initial-value problem over the time interval $[4, 14]$ with 1000 uniform time steps. About how many uniform time steps do we need to reduce the global error of our approximation by a factor of $\frac{1}{81}$?

Solution. Because the Runge-Kutta method is *fourth order*, its error scales like h^4. To reduce the error by a factor of $\frac{1}{81}$, we must reduce h by a factor of $(\frac{1}{81})^{\frac{1}{4}} = \frac{1}{3}$. Therefore we must increase the number of time steps by a factor of 3, which means that we need $3,000$ uniform time steps.

Remark. This problem is similar to some Group Work Exercises from last week.

(2) [4] Consider the initial-value problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 2u - u^2 \,, \qquad u(0) = 3 \,.$$

Approximate $u(.2)$ using one step of the Runge-midpoint method.
Leave your answer as an arithmetic expression.

Solution. Let $f(u) = 2u - u^2$. Set $u_0 = u(0) = 3$.
Then one step of the Runge-midpoint method with $h = .2$ yields

$$\begin{aligned}
f_0 &= f(u_0) = 2u_0 - u_0^2 && \text{evaluate } f(u) \text{ at step zero} \\
&\qquad = 2 \cdot 3 - 3^2 = 6 - 9 = -3 \,, \\
u_{\frac{1}{2}} &= u_0 + \tfrac{h}{2} f_0 && \text{half step by explicit Euler} \\
&= 3 + .1 \cdot (-3) = 3 - .3 = 2.7 \,, \\
f_{\frac{1}{2}} &= f(u_{\frac{1}{2}}) = 2u_{\frac{1}{2}} - u_{\frac{1}{2}}^2 && \text{evaluate } f(u) \text{ at the half step} \\
&\qquad = 2 \cdot 2.7 - (2.7)^2 \,, \\
u_1 &= u_0 + h f_{\frac{1}{2}} && \text{full step by Runge-midpoint} \\
&= 3 + .2\big[2 \cdot 2.7 - (2.7)^2\big] \,.
\end{aligned}$$

Therefore $u(.2) \approx u_1 = 3 + .2\big[2 \cdot 2.7 - (2.7)^2\big]$.

Remark. You do not have to evaluate the above arithemic expression for full credit. It evaluates to $u(.2) \approx u_1 = 2.622$.

Group Work Exercises for Problem 2 [5]

Let $u(t)$ be the solution of the initial-value problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 2u - u^2\,, \qquad u(0) = 3\,.$$

- Approximate $u(.2)$ using one step of the explicit Euler method.
- Approximate $u(.2)$ using two steps of the explicit Euler method.
- Approximate $u(.2)$ using one step of the Runge-trapezoidal method.
- The differential equation is autonomous. Use a phase-line portrait to describe how the solution $u(t)$ of the initial-value problem behaves.
  - Is $u(t)$ an increasing or decreasing function of t?
  - How does $u(t)$ behave as $t \to \infty$?

  Are the four numerical approximations consistent with this information?
- Find the explicit solution $u(t)$ of the initial-value problem analytically. Compute the exact value of $u(.2)$. How does this compare with the four approximations to it found by the numerical methods?

(3) [4] Determine if the following differential forms are exact or not. (Do not solve!)

(a) [2] $(y^2 - 3xy)\,\mathrm{d}x + (xy - x^2)\,\mathrm{d}y = 0\,.$

(b) [2] $(2xy - x^2)\,\mathrm{d}x + (y^2 + x^2)\,\mathrm{d}y = 0\,.$

Solution (a). This differential form is not exact because

$$\partial_y(y^2 - 3xy) = 2y - 3x \qquad \neq \qquad \partial_x(xy - x^2) = y - 2x\,.$$

Solution (b). This differential form is exact because

$$\partial_y(2xy - x^2) = 2x \qquad = \qquad \partial_x(y^2 + x^2) = 2x\,.$$

Group Work Exercises for Problem 3 [5]

- Find an integrating factor for $(y^2 - 3xy)\,\mathrm{d}x + (xy - x^2)\,\mathrm{d}y = 0\,.$
- Find an implicit general solution of $(y^2 - 3xy)\,\mathrm{d}x + (xy - x^2)\,\mathrm{d}y = 0\,.$
- Find an implicit general solution of $(2xy - x^2)\,\mathrm{d}x + (y^2 + x^2)\,\mathrm{d}y = 0\,.$
- Solve the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 3xy}{x^2 - xy}\,, \qquad y(1) = -2\,.$$

- Solve the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 - 2xy}{x^2 + y^2}\,, \qquad y(-1) = 1\,.$$

Remark. These initial-value problems lead directly to the differential forms in Problem 3.

Remark. The right-hand side of the first equation is undefined when $x = 0$ or $y = x$ and is differentiable elsewhere. The right-hand side of the second equation is undefined when $x = y = 0$ and is differentiable elsewhere.