Quiz 2 Solutions, Math 246, Professor David Levermore
Tuesday, 11 September 2018

(1) [6] Sketch the phase-line portrait for the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{(u-5)(u+1)^3(u+4)^2}{(u-2)^2}\,.$$

Classify each stationary point as being either stable, unstable, or semistable. (You do not have to find the solution!)

Solution. This equation is autonomous. Its right-hand side is undefined at $u = 2$ and is differentiable elsewhere. Its stationary points are found by setting

$$\frac{(u-5)(u+1)^3(u+4)^2}{(u-2)^2} = 0\,.$$

Therefore the stationary points are $u = -4$, $u = -1$, and $u = 5$. A sign analysis of the right-hand side shows that the phase-line portrait is

$$\begin{array}{ccccccccc} + & & + & & - & & - & & + \\ \to\to\to\to & \bullet & \to\to\to\to & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \circ & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \to\to\to\to \quad u \\ & -4 & & -1 & & 2 & & 5 & \\ & \text{semistable} & & \text{stable} & & \text{undefined} & & \text{unstable} & \end{array}$$

Remark. Here the terms stable, unstable, and semistable are applied to solutions. The point $u = 2$ is not a solution, so these terms should not be applied to it.

Group Work Exercises for Problem 1 [4]

- For what initial values $u(0)$ will $u(t) \to -4$ as $t \to \infty$?
- For what initial values $u(0)$ will $u(t) \to -1$ as $t \to \infty$?
- For what initial values $u(0)$ will $u(t) \to 5$ as $t \to \infty$?
- For what initial values $u(0)$ will $u(t)$ blow up in finite time as t increases?
- For what initial values $u(0)$ will $u'(t)$ blow up in finite time as t increases, but $u(t)$ will not blow up?
- If $u(0)$ lies within the interval $[-4, -1]$ then what is the interval of definition of the solution $u(t)$?

Remark. An implicit general solution can be found analytically, but with difficulty. An explicit general solution cannot be found analytically. Therefore a contour plot could be made of an implicit solution if you wanted more quantitative information about the solutions. But the phase-line portrait already gives enough information to address many questions!

(2) [4] In the absence of predators the population of mosquitoes in a certain area would increase at a rate proportional to its current population such that it would triple every five weeks. There are 200,000 mosquitoes in the area when a flock of birds arrives that eats 40,000 mosquitoes per week. Write down an initial-value problem that governs the population of mosquitoes in the area after the flock of birds arrives. (Do not solve the initial-value problem!)

Solution. Let $M(t)$ be the number of mosquitoes at time t weeks. Tripling every five weeks gives a growth rate r that satisfies $e^{r5} = 3$, whereby $r = \frac{1}{5}\log(3)$. Therefore the initial-value problem that M satisfies is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \tfrac{1}{5}\log(3)M - 40,000\,, \qquad M(0) = 200,000\,.$$

Group Work Exercises for Problem 2 [3]

- Will $M(t)$ be an increasing or decreasing function of t?
- Is the flock of birds large enough to control the mosquitoes?
- How do the answers to the previous two questions change if there were 180,000 mosquitoes in the area when the same flock of birds arrives?

Remark. In the above exercises it is helpful to know that $\frac{12}{11} < \log(3) < \frac{11}{10}$.

Group Work Exercises for Quiz 3 [3]

Suppose that we have used a numerical method to approximate the solution of an initial-value problem over the time interval $[5, 25]$ with 1000 uniform time steps. Suppose that we want to reduce the global error of our approximation by a factor of $\frac{1}{625}$.

- About how many time steps are needed if we are using the explicit Euler method?
- About how many time steps are needed if we are using a second-order method?
- About how many time steps are needed if we are using a fourth-order method?