

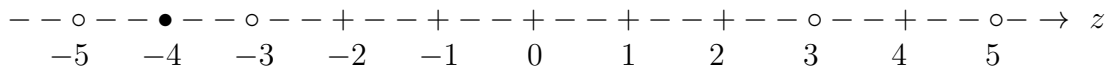
Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 4 September 2018

(1) [2] What is the interval of definition for the solution of the initial-value problem

$$\frac{dv}{dz} + \frac{\sin(z)}{z^2 - 9} v = \frac{\cos(z)}{z^2 - 25}, \quad v(-4) = 2.$$

(You do not need to solve the differential equation to answer this question!)

Solution. This is a nonhomogeneous linear equation that is already in normal form. The coefficient $\sin(z)/(z^2 - 9)$ is undefined at $z = \pm 3$ and is continuous elsewhere. The forcing $\cos(z)/(z^2 - 25)$ is undefined at $z = \pm 5$ and is continuous elsewhere. The initial time is $z = -4$. This can be pictured on the z -axis as follows.



Therefore the interval of definition for the solution is $(-5, -3)$ because:

- the initial time $z = -4$ is in $(-5, -3)$,
- the coefficient and forcing are both continuous over $(-5, -3)$,
- the forcing is undefined at $z = -5$,
- the coefficient is undefined at $z = -3$.

(2) [4] Solve the initial-value problem

$$t \frac{du}{dt} + 4u = 6t^2, \quad u(1) = 3.$$

Solution. This is a nonhomogeneous linear equation. Its normal form is

$$\frac{du}{dt} + \frac{4}{t} u = 6t.$$

The coefficient is undefined at $t = 0$ and is continuous elsewhere. The forcing is continuous everywhere. Because the initial time is $t = 1$, the interval of definition will be $(0, \infty)$.

An integrating factor is $e^{A(t)}$ where $A'(t) = 4/t$. Setting $A(t) = 4 \log(t)$ for $t > 0$, we obtain $e^{A(t)} = e^{4 \log(t)} = t^4$. Hence, the problem has the integrating factor form

$$\frac{d}{dt}(t^4 u) = t^4 \cdot (6t) = 6t^5.$$

Integrating both sides yields

$$t^4 u = t^6 + c.$$

Imposing the initial condition gives

$$1^4 \cdot 3 = 1^6 + c,$$

whereby $c = 1 \cdot 3 - 1 = 3 - 1 = 2$. Therefore the solution is

$$u = \frac{t^6 + 2}{t^4} = t^2 + \frac{2}{t^4}.$$

Group Work Exercises for Problems 1 and 2 [3]

- What is the interval of definition for the solution of the initial-value problem

$$\frac{dv}{dz} + \frac{\sin(z)}{z^2 - 9} v = \frac{\cos(z)}{z^2 - 25}, \quad v(2) = -4.$$

Give your reasoning.

- Solve the initial-value problem

$$t \frac{du}{dt} + 4u = 6t^2, \quad u(1) = -3.$$

Give the interval of definition for the solution.

- Solve the initial-value problem

$$t \frac{du}{dt} + 4u = 6t^2, \quad u(-1) = 3.$$

Give the interval of definition for the solution.

- (3) [4] Find an implicit solution of the initial-value problem

$$\frac{dy}{dx} = -\frac{e^x}{2y}, \quad y(0) = -2.$$

Solution. This is a nonautonomous, separable equation. It is undefined at $y = 0$ and is continuous elsewhere. It has no stationary points. Its separated differential form is

$$2y \, dy = -e^x \, dx,$$

whereby

$$\int 2y \, dy = - \int e^x \, dx.$$

Upon integrating both sides we find the implicit general solution

$$y^2 = -e^x + c.$$

The initial condition $y(0) = -2$ then implies that

$$(-2)^2 = -e^0 + c,$$

whereby $c = 4 + 1 = 5$. Therefore an implicit solution of the initial-value problem is

$$y^2 = -e^x + 5.$$

Group Work Exercises for Problems 3 [3]

- Find the explicit solution of the initial-value problem and give its interval of definition.
- How do $y(x)$ and $y'(x)$ behave as x approaches each endpoint of its interval of definition?
- Find the explicit solution of the initial-value problem

$$\frac{dy}{dx} = -\frac{e^x}{2y}, \quad y(0) = 3.$$

and give its interval of definition.

Group Work Exercises for Quiz 2 [4]

Consider the equation

$$\frac{du}{dt} = \frac{(u+5)^2(u+1)^3(7-u)}{u-3}.$$

Let $u_1(t)$ and $u_2(t)$ be the solutions of it that satisfy $u_1(2) = -3$ and $u_2(-1) = 5$. (You do not need to find these solutions!)

- Sketch the phase-line portrait for the equation.
- Classify each stationary point as being either stable, unstable, or semistable.
- Evaluate $\lim_{t \rightarrow \infty} u_1(t)$ and $\lim_{t \rightarrow \infty} u_2(t)$.
- Evaluate $\lim_{t \rightarrow \infty} (u_2(t) - u_1(t))$.