Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 4 September 2018

(1) [2] What is the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\sin(z)}{z^2 - 9}\, v = \frac{\cos(z)}{z^2 - 25}\,, \qquad v(-4) = 2\,.$$

(You do not need to solve the differential equation to answer this question!)

Solution. This is a nonhomogeneous linear equation that is already in normal form. The coefficient $\sin(z)/(z^2 - 9)$ is undefined at $z = \pm 3$ and is continuous elsewhere. The forcing $\cos(z)/(z^2 - 25)$ is undefined at $z = \pm 5$ and is continuous elsewhere. The initial time is $z = -4$. This can be pictured on the z-axis as follows.

$$\begin{array}{ccccccccccccc} --\circ-- & \bullet & --\circ-- & +-- & +-- & +-- & +-- & +-- & \circ-- & +-- & \circ- & \rightarrow & z \\ -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & & \end{array}$$

Therefore the interval of definition for the solution is $(-5, -3)$ because:

- the initial time $z = -4$ is in $(-5, -3)$,
- the coefficient and forcing are both continuous over $(-5, -3)$,
- the forcing is undefined at $z = -5$,
- the coefficient is undefined at $z = -3$.

(2) [4] Solve the initial-value problem

$$t\,\frac{\mathrm{d}u}{\mathrm{d}t} + 4u = 6t^2\,, \qquad u(1) = 3\,.$$

Solution. This is a nonhomogeneous linear equation. Its normal form is

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{4}{t}\,u = 6t\,.$$

The coefficient is undefined at $t = 0$ and is continuous elsewhere. The forcing is continuous everywhere. Because the initial time is $t = 1$, the interval of definition will be $(0, \infty)$.

An integrating factor is $e^{A(t)}$ where $A'(t) = 4/t$. Setting $A(t) = 4\log(t)$ for $t > 0$, we obtain $e^{A(t)} = e^{4\log(t)} = t^4$. Hence, the problem has the intgrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t^4 u\big) = t^4 \cdot (6t) = 6t^5\,.$$

Integrating both sides yields

$$t^4 u = t^6 + c\,.$$

Imposing the initial condition gives

$$1^4 \cdot 3 = 1^6 + c\,,$$

whereby $c = 1 \cdot 3 - 1 = 3 - 1 = 2$. Therefore the solution is

$$u = \frac{t^6 + 2}{t^4} = t^2 + \frac{2}{t^4}\,.$$

Group Work Exercises for Problems 1 and 2 [3]

- What is the interval of definition for the solution of the initial-value problem
$$\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\sin(z)}{z^2 - 9}\, v = \frac{\cos(z)}{z^2 - 25}\,, \qquad v(2) = -4\,.$$
Give your reasoning.
- Solve the initial-value problem
$$t\,\frac{\mathrm{d}u}{\mathrm{d}t} + 4u = 6t^2\,, \qquad u(1) = -3\,.$$
Give the interval of definition for the solution.
- Solve the initial-value problem
$$t\,\frac{\mathrm{d}u}{\mathrm{d}t} + 4u = 6t^2\,, \qquad u(-1) = 3\,.$$
Give the interval of definition for the solution.

(3) [4] Find an implicit solution of the initial-value problem
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{2y}\,, \qquad y(0) = -2\,.$$

Solution. This is a nonautonomous, separable equation. It is undefind at $y = 0$ and is continuous elsewhere. It has no stationary points. Its separated differential form is
$$2y\,\mathrm{d}y = -e^x\,\mathrm{d}x\,,$$
whereby
$$\int 2y\,\mathrm{d}y = -\int e^x\,\mathrm{d}x\,.$$
Upon integrating both sides we find the implicit general solution
$$y^2 = -e^x + c\,.$$
The initial condition $y(0) = -2$ then implies that
$$(-2)^2 = -e^0 + c\,,$$
whereby $c = 4 + 1 = 5$. Therefore an implicit solution of the initial-value problem is
$$y^2 = -e^x + 5\,.$$

Group Work Exercises for Problems 3 [3]

- Find the explicit solution of the initial-value problem and give is interval of definition.
- How do $y(x)$ and $y'(x)$ behave as x approaches each endpoint of its interval of definition?
- Find the explicit solution of the initial-value problem
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^x}{2y}\,, \qquad y(0) = 3\,.$$
and give is interval of definition.

Group Work Exercises for Quiz 2 [4]

Consider the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{(u+5)^2(u+1)^3(7-u)}{u-3}\,.$$

Let $u_1(t)$ and $u_2(t)$ be the solutions of it that satisfy $u_1(2) = -3$ and $u_2(-1) = 5$. (You do not need to find these solutions!)

- Sketch the phase-line portrait for the equation.
- Classify each stationary point as being either stable, unstable, or semistable.
- Evaluate $\lim_{t\to\infty} u_1(t)$ and $\lim_{t\to\infty} u_2(t)$.
- Evaluate $\lim_{t\to\infty} \big(u_2(t) - u_1(t)\big)$.